Popp measure and the intrinsic Sub-Laplacian

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Outline

Sub-Riemannian Geometry (Reminder from the 1. talk)

"Sub-Riemannian geometry models motions under non-holonomic constraints".

Definition

A Sub-Riemannian manifold (shortly: SR-m) is a triple $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ with:

- M is a smooth manifold (without boundary), $\dim M \geq 3$ and $\mathcal{H} \subset TM$ is a vektor distribution.
- $\mathcal{H}$ is bracket generating of rank $k < \dim M$, i.e.

$$\text{Lie}_x \mathcal{H} = T_x M$$

- $\langle \cdot, \cdot \rangle_x$ is a smoothly varying family of inner products on $\mathcal{H}_x$ for $x \in M$.

Question: *Can we assign "geometric operators" to such a structure similar to the Laplacian in Riemannian geometry?*

Regular Distribution

Let $\mathcal{H} \subset TM$ denote a distribution on M we define vector spaces depending on $q \in M$:

$$\mathcal{H}^1 := \mathcal{H}, \quad \text{and} \quad \mathcal{H}^{r+1} := \mathcal{H}^r + \left[\mathcal{H}^r, \mathcal{H}\right].$$

where

$$\left[\mathcal{H}^r, \mathcal{H}\right]_q = \text{span}\Big\{ \left[X, Y\right]_q \, : \, X_q \in \mathcal{H}^r_q \text{ and } Y_q \in \mathcal{H}_q \Big\}.$$

This gives a flag

$$\mathcal{H} = \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \mathcal{H}^{r+1} \subset \cdots$$

Remark: $\mathcal{H}$ bracket generating: $\forall\, q \in M$, $\exists\, \ell_q \in \mathbb{N}$ with $\mathcal{H}^{\ell_q}_q = T_q M$.

Definition

$\mathcal{H}$ is called regular, if the dimensions $\dim \mathcal{H}^r_q$ are independent of $q \in M$.

A non-regular distribution, (Martinet distribution)

Here is an example of a distribution which is not regular across a line:

On $M = \mathbb{R}^3$ with coordinates $q = (x, y, z)$ consider the vector fields:

$$X := \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z} \quad \textit{and} \quad Y := \frac{\partial}{\partial y}.$$

Then we have the Lie bracket:

$$[X, Y] = -y\frac{\partial}{\partial z}.$$

Therefore:

$$\mathcal{H}_q^2 = \text{span}\Big\{X, Y, y\frac{\partial}{\partial z}\Big\} \quad \text{in part.} \quad \mathcal{H}_{(x,0,z)}^2 = \text{span}\Big\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\Big\}.$$

Observe: The dimension of $\mathcal{H}_q^2$ "jumps":

$$\dim \mathcal{H}_q^2 = \begin{cases} 2, & \text{if } y = 0, \\ 3, & \text{if } y \neq 0. \end{cases}$$

From the Riemannian to the Subriemannian Laplacian

Goal

In analogy to the Laplace operator in Riemannian geometry we want to assign a Sub-Laplace operator to the Subriemannian structure.

1. Recall the definition of the Beltrami-Laplace operator:

Let (M, g) be an oriented Riemannian manifold with $\dim M = n$ and let $[X_1, \cdots, X_n]$ be a local orthonormal frame around a point $q \in M$.

Definition

The Riemannian volume form ω is defined through the requirement:

$$\omega(X_1, \cdots, X_n) = 1.$$

Or in coordinates:

$$\omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n \quad \textit{where} \quad g_{ij} = g(\partial_i, \partial_j), \ \partial_i := \frac{\partial}{\partial x_i}.$$

Divergence and gradient in Riemannian geometry

We review the definition of the Laplace operator in Riemannian geometry:

Gradient of a smooth function: Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be smooth:

$$\text{grad}(\varphi) = \left(\frac{\partial \varphi}{\partial x_1}, \cdots, \frac{\partial \varphi}{\partial x_n}\right) = J_\varphi = \text{"}\textit{total derivative.}\text{"}$$

Generalization to a Riemannian manifold (M, g): Let $\varphi \in C^\infty(M, \mathbb{R})$:

Gradient

The gradient $\text{grad}(\varphi)$ of the function φ is the unique vector field with

$$g_q\big(\text{grad}(\varphi), v\big) = d\varphi(v), \qquad \forall q \in M, \ \forall v \in T_qM.$$

Here is a useful formula:

Lemma: *Let* $[X_1, \cdots, X_n]$ *be a local orthonormal frame around* $q \in M$*:*

$$\text{grad}\,(\varphi) = \sum_{i=1}^n X_i(\varphi) \cdot X_i \qquad \textit{around } q.$$

Divergence of a vector field:

Let X be a vector field on M and $\mathcal{L}_X$ the Lie derivative in direction X.

Definition: Define the divergence of X through the equation:

$$\mathcal{L}_X \omega = \text{div}_\omega(X) \cdot \omega. \qquad (*)$$

divergence = "*point-wise constant of proportionality*".

Lie derivative (reminder)

Here $\mathcal{L}_X$ denotes the Lie derivative along X of a differential form:

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X \qquad \text{(Cartan's formula)}.$$

In case of a volume form ω we have $d\omega = 0$ and therefore

$$\mathcal{L}_X \omega = d\big(\iota_X \omega\big) = \text{div}_\omega(X) \cdot \omega.$$

Observation:

In $(*)$ we need not necessarily choose the Riemannian volume form ω!

Divergence (interpretation)

The divergence of a vector field X - roughly speaking - measures how much the flow X changes the volume:

Let X be a vector field on M and $\Omega \subset M$ be compact. For a time $t > 0$ sufficiently small consider

$$e^{tX} : \Omega \to M \qquad \text{(flow of } X\text{)}.$$

Divergence and the "Change of volume"

$$\frac{d}{dt}\Big|_{t=0} \int_{e^{tX}(\Omega)} \omega = -\int_{\Omega} \operatorname{div}_{\omega}(X)\omega.$$

Definition (Laplace operator)

Let ω be the Riemannian volume form on (M, g). The Laplace operator Δ acting on smooth functions $\varphi \in C^{\infty}(M)$ is defined by :

$$\Delta\varphi = \operatorname{div}_{\omega} \circ \operatorname{grad}(\varphi).$$

Let $[X_1, \cdots, X_n]$ be a local orthonormal frame. We can use the previous expression of the gradient:

$$\begin{aligned}\Delta\varphi &= \operatorname{div}_{\omega} \circ \operatorname{grad}(\varphi)\\ &= \operatorname{div}_{\omega}\left(\sum_{i=1}^{n} X_i(\varphi) \cdot X_i\right).\end{aligned}$$

and use the rule $\operatorname{div}_{\omega}(f \cdot X) = Xf + f\operatorname{div}_{\omega}(X)$ where X is a vector field and f a function on M:

$$\Delta(\varphi) = \sum_{i=1}^{n} \Big[X_i^2(\varphi) + \operatorname{div}_{\omega}(X_i) \cdot X_i(\varphi)\Big].$$

Laplace-Beltrami operator on (M, g)

Lemma

The Laplacian on (M, g) acting on functions has the form:

$$\Delta = \underbrace{\sum_{i=1}^{n} X_i^2}_{\text{2nd order}} + \underbrace{\sum_{i=1}^{n} \mathrm{div}_\omega(X_i) \cdot X_i}_{\text{first order}}.$$

Observation:

- The volume form ω only appears in the first order part.
- Δ is independent of the choice of orthonormal frame $[X_1, \cdots, X_n]$.

Remark: The Laplace operator Δ appears in the heat equation on M

$$\frac{\partial}{\partial t} - \Delta = 0,$$

modeling the diffusion of the temperature on a body. On the other hand:

Heat diffusion should be influenced by the geometry of the object.

The Sub-Laplacian

Idea: *Use the same strategy to assign a second order differential operator to a sub-Riemannian manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ with regular distribution $\mathcal{H}$.* [1]

We need:

- *"Subriemannian gradient"*
- *"Subriemannian divergence"*

Horizontal gradient and ω-divergence

Let ω be a smooth measure, X a vector field on M and $\varphi \in C^\infty(M)$:

$$\mathcal{L}_X(\omega) = \mathrm{div}_\omega(X)\,\omega \qquad (\omega\text{-divergence})$$

$$\Big\langle \underbrace{\mathrm{grad}_{\mathcal{H}}(\varphi)}_{\in \mathcal{H}_q}, v \Big\rangle_q = d\varphi(v), \quad v \in \mathcal{H}_q \qquad \text{(horizontal gradient)}.$$

These equations - together with the horizontality condition of the gradient - define div_ω and $\mathrm{grad}_{\mathcal{H}}$.

[1] based on: A. Agrachev, U. Boscain, J.-P. Gauthier, F. Rossi, *The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups*,

Sub-Laplacian

Definition

The Sub-Laplacian on a SR-manifold $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ associated to a smooth volume ω is defined by

$$\Delta_{\text{sub}} := \text{div}_\omega \circ \text{grad}_{\mathcal{H}}.$$

Consider a local orthonormal frame for $\mathcal{H}$

$$[X_1, \cdots, X_m] \qquad \textit{with} \qquad m \leq n = \dim M.$$

Similar to the Laplacian we can express Δ_{sub} in the form:

$$\Delta_{\text{sub}} = \sum_{i=1}^{m} \Big[X_i^2 + \text{div}_\omega(X_i) \cdot X_i \Big].$$

The Sub-Laplacian

Theorem

The Subriemannian Laplacian associated to a smooth measure ω is negative, symmetric and, if M is compact, essentially self-adjoint on $C_c^\infty(M) \subset L^2(M)$.

Proof: Let $f \in C_c^\infty(M)$ and let X be a vector field on M. One shows:

$$\int_M f \cdot \text{div}_\omega(X)\, \omega = -\int_M X(f)\, \omega = -\int_M df(X)\, \omega.$$

Choose $X = \text{grad}_{\mathcal{H}}(g)$ with $g \in C_c^\infty(M)$. Symmetry and negativity follow:

$$\int_M f \cdot (\Delta_{\text{sub}} g)\, \omega = -\int_M \langle \text{grad}_{\mathcal{H}} f, \text{grad}_{\mathcal{H}} g \rangle \omega.$$

Essentially selfadjointness is shown in.[2] □

[2] R. Strichartz, *Sub-Riemannian Geometry*, J. Differential Geom. 24, (1986), 221-263.

The Hausdorff volume and Popp volume

Question: *How to choose the smooth measure ω in the ω-divergence?*

Requirement

If we would like to have a "geometric operator" such measure should only depend on the internal data of the SR-structure.

Possible candidates:

- The Hausdorff measure of (M, d_{cc})? (next slides)
- The Popp measure on $\mathcal{P}$ (next slides). This measure is a-priori smooth by construction.

Remark

- Maybe both measures coincide?
- If we have a "canonical measure" ω we may consider the sub-Riemannian heat equation:

$$\partial_t - \Delta_{\text{sub}} = 0$$

  and study its geometric significance in comparison with the Riemannian setting.

Hausdorff measure

Let (M, d) be a metric space and $\Omega \subset M$. Let

- $\varepsilon, s > 0$,
- $\{U_\alpha\}_\alpha$ a covering of Ω by open sets.

Consider:

$$\mu^s_\varepsilon(\Omega) := \inf \Big\{ \sum_\alpha \big[\text{diam } U_\alpha\big]^s \ : \ \forall \alpha : \ \text{diam } U_\alpha < \varepsilon \Big\}.$$

Hausdorff measure

The value

$$\mu^s(\Omega) := \lim_{\varepsilon \to 0} \mu^s_\varepsilon(\Omega) \in [0, \infty) \cup \{\infty\}$$

is called s-dimensional Hausdorff measure of Ω.

Proposition: *There is a unique value Q, the Hausdorff dimension of Ω, with $\mu^s(\Omega) = \infty$ for $s < Q$ and $\mu^s(\Omega) = 0$ for $s > Q$.*

Hausdorff measure

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Sub-Riemannian manifold. Then (M, d_{cc}) is a metric space with:

$$d_{cc}(A,B) = \inf \Big\{ \underset{\substack{\uparrow \\ \text{SR- length of } \gamma}}{L_{\text{SR}}(\gamma)} \, : \, \gamma(0) = A \quad \gamma(1) = B, \quad \gamma \text{ horizontal} \Big\}$$

$$= \textit{Carnot-Carathéodory distance on } M.$$

Definition

Let μ_{Haus}^{Q} be the Hausdorff measure of the metric space (M, d_{cc}).

Problem:

- it is hard to calculate μ_{Haus}^{Q} in general.
- not clear whether (or in which cases) the Hausdorff measure is a smooth measure on M.

Nilpotentization and Popp measure

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a regular Sub-Riemannian manifold.

Consider again the flag induced by the bracket generating distribution $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \mathcal{H}^{r+1} \subset \cdots$$

Notation: By definition $\dim \mathcal{H}_q^r$ for all r are independent of $q \in M$, where:

$$\mathcal{H}^1 := \mathcal{H} = \textit{"sheave of smooth horizontal vector fields"},$$
$$\mathcal{H}^{r+1} := \mathcal{H}^r + [\mathcal{H}^r, \mathcal{H}],$$

with

$$[\mathcal{H}^r, \mathcal{H}]_q = \text{span}\Big\{ [X, Y]_q \, : \, X_q \in \mathcal{H}_q^r \text{ and } Y_q \in \mathcal{H}_q \Big\}$$

Nilpotentization

For each $q \in M$ we obtain a graded vector space:

$$\begin{aligned} \mathrm{gr}(\mathcal{H})_q &= \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q \oplus \cdots \oplus \mathcal{H}_q^r/\mathcal{H}_q^{r-1} \\ &= \text{nilpotentization.} \end{aligned}$$

Observations:

(a) Lie brackets of vector fields on M induce a Lie algebra structure on $\mathrm{gr}(\mathcal{H})_q$ *(respecting the grading)*.

(b) The Lie algebra in (a) is nilpotent, i.e. there is $n \in \mathbb{N}$ such that:

$$\Big[X_1\big[X_2 \cdots \big[X_n, X\big] \cdots \big]\Big] = 0, \qquad \forall\ X_1, \cdots, X_n, X \in \mathfrak{g}. \qquad (*)$$

The minimal n in (b) is called the step of the nilpotent Lie algebra.

Example: *The step of the nilpotentization* $\mathrm{gr}(\mathcal{H})_q$ *is* r.

Popp measure: construction in the case $r = 2$

$(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ = regular Sub-Riemannian manifold. Let $r = 2$ and $q \in M$.

1. step: Let $v, w \in \mathcal{H}_q$ and V, W be horizontal vector fields near p with:

$$V(q) = v \quad \textit{and} \quad W(q) = w.$$

Consider the map

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \bmod \mathcal{H}_q.$$

Some properties of π:

- π is surjective,
- the inner product on $\mathcal{H}_q$ induces an inner product on $\mathcal{H}_q \otimes \mathcal{H}_q$.

Question: *Is the map π well-defined?*

With horizontal vector fields V and W with $V(q) = v$ and $W(q) = w$:

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \text{ mod } \mathcal{H}_q.$$

Lemma

The map π is independent of the choice of V and W.

Proof.

Let $\widetilde{V}$ and $\widetilde{W}$ be different horizontal extensions of v and w, i.e.

$$\widetilde{V}(p), \widetilde{W}(p) \in \mathcal{H}_p \quad \forall\, p \in M, \qquad \textit{and} \qquad \widetilde{V}(q) = v \;\; \widetilde{W}(q) = w.$$

With a local frame $[X_1, \cdots, X_m]$ of $\mathcal{H}$ we can write:

$$\widetilde{V} = V + \sum_{i=1}^{m} f_i X_i \quad \textit{and} \quad \widetilde{W} = W + \sum_{i=1}^{m} g_i X_i,$$

where $f_i, g_i \in C^\infty(M)$ fulfill $f_i(q) = g_i(q) = 0$. □

Proof: (continued)

We form Lie brackets:

$$\big[\widetilde{V}, \widetilde{W}\big] = \Big[V + \sum_{i=1}^{m} f_i X_i, W + \sum_{j=1}^{m} g_j X_j\Big]$$

$$= [V, W] + \sum_{j=1}^{m} \big[V, g_j X_j\big] - \big[W, f_j X_j\big] + \sum_{i,j=1}^{m} \big[f_i X_i, g_j X_j\big] = (*).$$

We use the rule:

$$\big[V, g_j X_j\big] = V(g_j) X_j + g_j \big[V, X_j\big]$$

to obtain $[f_i X_i, g_j X_j]_q = 0$ and

$$\big[\widetilde{V}, \widetilde{W}\big]_q = [V, W]_q + \sum_{j=1}^{m} \Big[V(g_j) - W(f_j)\Big] X_j(q) = \big[V, W\big]_q \text{ mod } \mathcal{H}_q.$$

Summary

Consider the map

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := \big[V, W\big]_q \bmod \mathcal{H}_q.$$

Some properties of π:

- π is surjective,
- the inner product on $\mathcal{H}_q$ induces an inner product on $\mathcal{H}_q \otimes \mathcal{H}_q$.
- Finally:

$$\mathcal{H}_q^2/\mathcal{H}_q \cong \mathrm{kern}(\pi)^\perp \subset \mathcal{H}_q \otimes \mathcal{H}_q.$$

Consequence: The inner product on $\mathcal{H}_q$ induces a inner product on:

$$\mathrm{gr}(\mathcal{H})_q = \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q = \textit{nilpotentization}.$$

This inner product induces a canonical volume form, i.e. an element:

$$\mu_q \in \Lambda^n \mathrm{gr}(\mathcal{H})_q^* \cong \big(\Lambda^n \mathrm{gr}(\mathcal{H})_q\big)^*.$$

Definition: Popp measure ($r = 2$)

2. step: *We need to produce a volume form on M itself.*

Let $n = \dim M$. Then there is a canonical isomorphism

$$\Theta_q : \Lambda^n(T_qM) \to \Lambda^n \mathrm{gr}(\mathcal{H})_q.$$

Explicitly:

Let $v_1, \cdots, v_n$ be a basis of T_qM s. t.$v_1, \cdots, v_m$ is a basis of $\mathcal{H}_q$. Put

$$\Theta_q\big(v_1 \wedge \cdots \wedge v_n\big) := v_1 \wedge \cdots \wedge v_m \,\widehat{\otimes}\, \Big(v_{m+1} + \mathcal{H}_q\Big) \wedge \cdots \wedge \Big(v_n + \mathcal{H}_q\Big).$$

Then Θ_q is independent of the choice of such basis.

Definition: Popp measure

With the volume form $\mathcal{P}_q := \Theta_q^*(\mu_q) = \mu_q \circ \Theta_q \in (\Lambda^n T_qM)^*$ we form

$$\mathcal{P} \in \Omega^n(M) = \textit{Popp measure}.$$

Remark: $\mathcal{P}$ *generalizes to SR-structures of arbitrary step $r > 0$.*

Intrinsic Sub-Laplacian

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a regular SR-manifold with Popp measure $\mathcal{P}$.

Definition

The intrinsic Sub-Laplacian on M is the Sub-Laplacian associated to $\mathcal{P}$:

$$\Delta_{\mathcal{P}} = \text{div}_{\mathcal{P}} \circ \text{grad}_{\mathcal{H}}.$$

1. Example: Martinet distribution

Consider the Martinet distribution on $\mathbb{R}^3$:

Define vector fields:

$$X := \frac{\partial}{\partial x} + \frac{y^2}{2}\frac{\partial}{\partial z}, \qquad Y := \frac{\partial}{\partial y} \quad \textit{and} \quad Z = \frac{\partial}{\partial z}.$$

Consider the following distribution: With $q \in \mathbb{R}^3$ put:

$$\begin{aligned} \mathcal{H}_q :&= \text{span}\Big\{X_q, Y_q\Big\} \\ &= \text{kern}\,(\Theta_q) \quad \textit{where} \quad \Theta = dz - \frac{y^2}{2}dx \\ &= \text{Martinet distribution}. \end{aligned}$$

- An inner product on $\mathcal{H}_q$ is defined by declaring X_q and Y_q orthonormal.
- **bracket relations:**

$$\Big[X, Y\Big] = -yZ \quad \textit{and} \quad \Big[Y, \big[X, Y\big]\Big] = Z.$$

1. Example: Martinet distribution (continued)

The Martinet distribution $\mathcal{H}$ is

- bracket generating on $\mathbb{R}^3$ and of step 3 if $y = 0$,
- regular of step 2 restricted to

$$M_{y=0} := \{(x, y, z)^t \ : \ y \neq 0\}.$$

Popp measure on $M_{y=0}$: Consider the map:

$$\pi : \mathcal{H}_q \otimes \mathcal{H}_q \to \mathcal{H}_q^2/\mathcal{H}_q : \pi(v \otimes w) := [X, Y]_q \bmod \mathcal{H}_q,$$

where $v = X_q$ and $w = Y_q$. Then:

$$\begin{aligned}(\ker \pi)^\perp &= \operatorname{span}\Big\{X \otimes X, Y \otimes Y, \frac{1}{\sqrt{2}}(X \otimes Y + Y \otimes X)\Big\}^\perp \\ &= \operatorname{span}\Big\{\frac{1}{\sqrt{2}}(X \otimes Y - Y \otimes X)\Big\}.\end{aligned}$$

1. Example: Martinet distribution (continued)

Using $[X, Y] = -yZ$ we find:

$$\begin{aligned}\frac{1}{\sqrt{2}}\pi\big[X \otimes Y - Y \otimes X\big] &= \sqrt{2} \cdot [X, Y] + \mathcal{H}_q \\ &= -\sqrt{2}yZ + \mathcal{H}_q.\end{aligned}$$

This induces an inner product norm on $\mathcal{H}_q^2/\mathcal{H}_q = \operatorname{span}\{Z\} + \mathcal{H}_q$ via:

$$\|Z + \mathcal{H}_q\|_q = \frac{1}{\sqrt{2}|y|}.$$

Take the dual basis to $[X, Y, \sqrt{2}|y|Z]$ which is

$$\Big[X^* = dx, Y^* = dy, (\sqrt{2}|y|Z)^* = (\sqrt{2}|y|)^{-1}(dz - \frac{y^2}{2}dx)\Big].$$

Popp measure:

$$\mathcal{P} = X^* \wedge Y^* \wedge (\sqrt{2}|y|Z)^* = \frac{1}{\sqrt{2}|y|} dx \wedge dy \wedge dz.$$

Intrinsic Sub-Laplacian for the Martinet distribution

Knowing the Popp measure, we can calculate the intrinsic Sub-Laplacian

$$\Delta_{\text{sub}} = \text{div}_{\mathcal{P}} \circ \text{grad}_{\mathcal{H}} \quad \text{on} \quad M_{y=0} \subset \mathbb{R}^3.$$

Recall the following explicit expression:

$$\Delta_{\text{sub}} = X^2 + Y^2 + \text{div}_{\mathcal{P}}(X)\, X + \text{div}_{\mathcal{P}}(Y)\, Y.$$

Note that

$$\text{div}_{\mathcal{P}}(X) \cdot \mathcal{P} = \mathcal{L}_X \mathcal{P} = d\big(\iota_X \mathcal{P}\big) = d\left(\frac{1}{\sqrt{2}|y|} dy \wedge dz\right) = 0 \cdot \mathcal{P},$$

$$\text{div}_{\mathcal{P}}(Y) \cdot \mathcal{P} = \mathcal{L}_Y \mathcal{P} = d\big(\iota_Y \mathcal{P}\big) = -d\left(\frac{1}{\sqrt{2}|y|} dx \wedge dz\right) = -\frac{1}{y} \cdot \mathcal{P}.$$

Instrinsic Sub-Laplacian

The intrinsic Sub-Laplacian becomes singular at the $y = 0$-surface.

$$\Delta_{\text{sub}} = X^2 + Y^2 - \frac{1}{y}\, Y.$$

2. Example

Question: *Is the Popp measure a multiple of the Hausdorff measure on (M, d_{cc})? Is the Hausdorff measure smooth?*

In general the answer is unknown!

Example

Let G be a nilpotent Lie group with left-invariant SR-structur, e.g.

$$G = \mathbb{H}_3 = \text{Heisenberg group of dimension three.}$$

Then one can show:

$$\mathcal{P} = \text{Popp measure} \quad \text{and} \quad \mu^Q_{\text{Hau}} = \text{Hausdorff measure}$$

are left-invariant. Hence:

$$\alpha \cdot \mathcal{P} = \mu_{\text{Haar}} = \textit{Lebesgue measure} = \beta \cdot \mu^Q_{\text{Hau}}.$$

The constant β is unknown.

Popp measure and local isometries

Riemannian isometry:

This is a *diffeomorphism* with differential being an isometry for the Riemannian metric.

Definition (volume preserving transformation)

Let M be a manifold and $\mu \in \Omega^n(M)$ a volume form. A diffeomorphism $\Phi : M \to M$ is a volume preserving transformation if

$$\phi^* \mu = \mu.$$

Standard fact:

Riemannian isometries are volume preserving transformation for the Riemannian volume.

Question: Is there an analogous statement in the case of a Sub-Riemannian manifold and the Popp measure?

Sub-Riemannian isometries

Let $(M, \mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Sub-Riemannian manifold and

$$\Phi : M \to M \qquad (*)$$

a diffeomorphism.

Definition

The map $(*)$ is called isometry, if its differential $\Phi_* : TM \to TM$ preserves the Sub-Riemannian structure, i.e.

- $\Phi_*(\mathcal{H}_q) = \mathcal{H}_{\Phi(q)}$ for all $q \in M$,
- For all $q \in M$ and all horizontal vector fields X, Y:

$$\langle \Phi_* X, \Phi_* Y \rangle_{\Phi(q)} = \langle X, Y \rangle_q.$$

We write $\text{Iso}(M)$ for the group of all isometries on the SR-manifold M.

Popp volume and isometries

Theorem (D. Barilari, L. Rizzi, 2012)

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular Sub-Riemannian manifold

(a) *Sub-Riemannian isometries are volume preserving for Popp's volume.*

(b) *If $\text{Iso}(M)$ acts transitively, then Popp's volume is the unique volume (up to multiplication by a constant) with (a).*

Example

Let $M = G$ be a Lie group with a left-invariant SR-structure. Then the left-translation

$$L_g : G \to G : h \mapsto L_g h = g * h$$

obviously defines an isometry.

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