Sub-Laplacian and the heat equation

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Outline

1. Sub-Laplacian on nilpotent Lie groups
2. Nilpotent approximation
3. Sub-elliptic heat kernel asymptotic

Intrinsic Sub-Laplacian (Reminder from the 2nd talk)

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular SR-manifold with Popp measure $\mathcal{P}$.

Definition

The intrinsic Sub-Laplacian on M is the Sub-Laplacian associated to $\mathcal{P}$:

$$\Delta_{\text{sub}} = \text{div}_{\mathcal{P}} \circ \text{grad}\ _{\mathcal{H}}$$

where (with the Lie derivative $\mathcal{L}_X$)

$$\mathcal{L}_X \mathcal{P} = \text{div}_{\mathcal{P}}(X) \cdot \mathcal{P} \qquad \textit{and} \qquad \text{grad}_{\mathcal{H}} = \textit{horizontal gradient.}$$

Here: $\mathcal{P}=$ Popp measure and

$$\Big\langle \underbrace{\text{grad}_{\mathcal{H}}(\varphi)}_{\in \mathcal{H}_q}, v \Big\rangle_q = d\varphi(v), \quad v \in \mathcal{H}_q \qquad \textit{(horizontal gradient).}$$

The Sub-Laplacian on nilpotent Lie groups

Carnot group

A Carnot group is a connected, simply connected Lie group G, with Lie algebra $\mathfrak{g}$ allowing a stratification

$$\mathfrak{g} = V_1 \oplus \cdots \oplus V_r.$$

Moreover, the following bracket relations respecting the stratification hold:

$$[V_1, V_j] = V_{j+1}, \quad j = 1, \cdots, r-1,$$
$$[V_j, V_r] = \{0\}, \quad j = 1, \cdots, r.$$

In particular $\mathfrak{g}$ is nilpotent of step r.

Example: *Let $\mathfrak{h}_3$ be the Heisenberg Lie algebra. Then*

$$\mathfrak{h}_3 = \text{span}\{X, Y\} \oplus \text{span}\{Z\},$$

where $[X, Y] = Z$. This is a 2-step case.

Two classical results

Theorem (Lie's third theorem)

Every finite dimensional real Lie algebra is the Lie algebra of a Lie group.

Recall that a Lie group homomorphism is a smooth group isomorphism between Lie groups.

Theorem

Let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let

$$\Phi : \mathfrak{g} \to \mathfrak{h}$$

denote a Lie algebra homomorphism. If G is simply connected, then there is a unique Lie group homomorphism

$$f : G \to H$$

such that $\Phi = df$ (the differential of f).

Carnot group

A combination of the last theorem gives:

Corollary

For every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ there is a simply connected Lie group G which has $\mathfrak{g}$ as Lie algebra. Moreover, G is unique up to isomorphisms.

This leads to the notion of Carnot group.

Definition

Let $\mathfrak{g}$ be a Carnot Lie algebra. The connected, simply connected Lie group G (up to isomorphisms) with Lie algebra $\mathfrak{g}$ is called Carnot group.

Remark: If $\mathfrak{g}$ has step r, we call the Carnot group G of step r.

Example: Engel group

Consider the Engel group $\mathcal{E}_4 \cong \mathbb{R}^4$ as a matrix group:

$$\mathcal{E}_4 = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2} & z \\ 0 & 1 & x & w \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \; : \; x, y, w, z \in \mathbb{R} \right\} \subset \mathbb{R}^{4\times 4}.$$

Then $\mathcal{E}_4$ has the Lie algebra $\mathfrak{e}_4$ with non-trivial bracket relations:

$$[X, Y] = W \qquad \text{und} \qquad \big[X, \underbrace{[X, Y]}_{=W}\big] = Z$$

and stratification

$$\mathfrak{e}_4 = \text{span}\{X, Y\} \oplus \text{span}\{W\} \oplus \text{span}\{Z\}.$$

Corollary

The Engel group $\mathcal{E}_4$ is a Carnot group of step 3.

Nilpotent approximation

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular Sub-Riemannian manifold.

Consider again the flag induced by the bracket generating distribution $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots \subset \mathcal{H}^r \subset \mathcal{H}^{r+1} \subset \cdots$$

Notation: By definition $\dim \mathcal{H}_q^r$ for all r are independent of $q \in M$, where:

$$\begin{aligned} \mathcal{H}^1 &:= \mathcal{H} = \text{"}\textit{sheave of smooth horizontal vector fields}\text{"}, \\ \mathcal{H}^{r+1} &:= \mathcal{H}^r + \big[\mathcal{H}^r, \mathcal{H}\big], \end{aligned}$$

with

$$\big[\mathcal{H}^r, \mathcal{H}\big]_q = \text{span}\Big\{ \big[X, Y\big]_q \; : \; X_p \in \mathcal{H}_p^r \textit{ and } Y_p \in \mathcal{H}_p \Big\}.$$

Nilpotent approximation

For each $q \in M$ we obtain a graded vector space:

$$\begin{aligned} \mathrm{gr}(\mathcal{H})_q &= \mathcal{H}_q \oplus \mathcal{H}_q^2/\mathcal{H}_q \oplus \cdots \oplus \mathcal{H}_q^r/\mathcal{H}_q^{r-1} \\ &= \text{nilpotentization.} \end{aligned}$$

Observations:

- Lie brackets of vector fields on M induce a Lie algebra structure on $\mathrm{gr}(\mathcal{H})_q$. *(respecting the grading).*
- Let $\mathrm{Gr}(\mathcal{H})_q$ denote the connected, simply connected nilpotent Lie group with Lie algebra $\mathrm{gr}(\mathcal{H})_q$.
- The space $\mathcal{H}_q \subset \mathrm{gr}(\mathcal{H})_q$ induces for each $q \in M$ a (left-invariant) SR-structure on the group $\mathrm{Gr}(\mathcal{H})_q$ (Example of talk 1).

Definition

The group $\mathrm{Gr}(\mathcal{H})_q$ with the induced SR-structure is called nilpotent approximation [a] of the SR-structure M at $q \in M$.

[a] It plays the role of a tangent space in Riemannian geometry

Nilpotent approximation

Conclusion:

Carnot groups seem to be a good local model of the SR-manifold. It may be helpful to understand the Sub-Laplacian and sub-elliptic heat flow on such groups.

Question

What is the intrinsic Sub-Laplacian on a Carnot group or (more generally) on any nilpotent Lie group?

Exponential coordinates: Let $(G, *)$ be a connected, simply connected nilpotent Lie group of dimension $\dim G = n$ and with Lie algebra $\mathfrak{g}$. Then

$$\exp : \mathfrak{g} \to G$$

is a diffeomorphism. Hence we can pullback the product on G to $\mathfrak{g} \cong \mathbb{R}^n$ via exp *(exponential coordinates)*.

Exponential coordinates

We have an identification:

$$(G, *) \cong (\mathfrak{g} \cong \mathbb{R}^n, \circ),$$

where

$$g \circ h := \log\Big(\exp(g) * \exp(h)\Big), \quad \text{for all} \quad g, h \in \mathfrak{g}.$$

Baker-Campbell-Hausdorff formula

Let $g, h \in \mathfrak{g}$, then

$$\exp(g) * \exp(h) =$$
$$= \exp\Big(g + h + \frac{1}{2}[g,h] + \frac{1}{12}\big[g,[g,h]\big] - \frac{1}{12}\big[h,[g,h]\big] \mp \cdots\Big)$$

Note: if $\mathfrak{g}$ is nilpotent, then the sum in the exponent is always finite.

Exponential coordinates

Using this formula above gives:

$$g \circ h = g + h + \frac{1}{2}[g,h] + \frac{1}{12}\big[g,[g,h]\big] - \frac{1}{12}\big[h,[g,h]\big] \mp \cdots \text{(finite)}.$$

Example

Consider the case $r = \text{step}\,\mathfrak{g} = 2$ and choose a decomposition

$$\mathfrak{g} = V_1 \oplus V_2$$

such that

$$[V_1, V_1] = V_2 \quad \textit{and} \quad [V_1, V_2] = [V_2, V_2] = 0.$$

Consider the SR-structure on $\mathfrak{g} \cong G$ defined by:

$$\mathcal{H} = V_1 = \text{"left-invariant vector fields."}$$

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Consider an inner product $\langle \cdot, \cdot \rangle$ on V_1 and chose an orthonormal basis:

$$[X_1, \cdots, X_m] = \text{"}orthonormal\ basis\ of\ V_1\text{"}.$$

Chose a basis $[Y_{m+1}, \cdots, Y_n]$ of V_2. Then there are structure constants c_{ij}^k such that

$$[X_i, X_j] = \sum_{\ell=m+1}^{n} c_{ij}^{\ell} Y_\ell, \qquad [X_i, Y_\ell] = 0 = [Y_\ell, Y_h].$$

This choice of basis gives a concrete identification $\mathfrak{g} \cong \mathbb{R}^n$.

Goal: *Calculate the left-invariant vector fields corresponding to the basis elements X_i.*

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Let $f \in C^\infty(\mathbb{R}^n)$ and $g = \sum_{j=1}^m x_j X_j \in \mathfrak{g}$. Then

$$\begin{aligned}
\big[X_i f\big](g) &= \frac{d}{dt} f\Big(g \circ tX_i\Big)_{|t=0} \\
&= \frac{d}{dt} f\Big(g + tX_i + \frac{1}{2}\big[g, tX_i\big]\Big)_{|t=0} \\
&= \frac{d}{dt} f\Big(g + tX_i + \frac{t}{2}\sum_{j=1}^m x_j \big[X_j, X_i\big]\Big)_{|t=0} \\
&= \frac{d}{dt} f\Big(g + tX_i + \frac{t}{2}\sum_{j=1}^m \sum_{\ell=m+1}^{n} x_j c_{ji}^{\ell} Y_\ell\Big)_{|t=0} \\
&= \left\{ \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{j=1}^m \sum_{\ell=m+1}^{n} x_j c_{ij}^{\ell} \frac{\partial}{\partial y_\ell} \right\} f(g).
\end{aligned}$$

Sub-Laplacian on nilpotent Lie groups

Example (continued)

We can identify $X_i \in V_1 \subset \mathfrak{g}$ with the following left-invariant vector field on $G \cong \mathbb{R}^n$:

$$\widetilde{X}_i := \frac{\partial}{\partial x_i} - \frac{1}{2}\sum_{j=1}^{m}\sum_{\ell=m+1}^{n} x_j c_{ij}^{\ell} \frac{\partial}{\partial y_\ell}.$$

Observations:

- the coefficients in front of $\frac{\partial}{\partial x_i}$ is one for $i = 1, \cdots, m$,
- in the double sum the variable x_i does not appear ($c_{ii}^{\ell} = 0$ for all ℓ).

Let $\mathcal{P}$ = Lebesgue measure be the Popp measure on $G \cong \mathbb{R}^n$.

Goal: *Calculate the $\mathcal{P}$-divergence of X_i for $i = 1, \cdots, m$:*

From the above observations:

$$\mathcal{L}_{\widetilde{X}_i}\Big(dx_1 \wedge \cdots \wedge dx_m \wedge dy_{m+1} \wedge \cdots \wedge dy_{n-m}\Big) = d \circ \iota_{\widetilde{X}_i}\mathcal{P} = d\Big(\mathcal{P}(\widetilde{X}_i, \cdot)\Big) = 0.$$

Therefore $\mathrm{div}_{\mathcal{P}}(X_i) = 0$ for all $i = 1, \cdots, m$.

Sub-Laplacian on nilpotent Lie groups

Example (continued)

Conclusion: In the above example of a step-2 nilpotent Lie group we have found:

$$\Delta_{\mathrm{sub}} = \sum_{i=1}^{m}\Big[\widetilde{X}_i^2 + \underbrace{\mathrm{div}_\omega(X_i)}_{=0} X_i\Big] = \sum_{i=1}^{m}\widetilde{X}_i^2.$$

Hence, the intrinsic sub-Laplacian has no first order terms. We say:

$$\Delta_{\mathrm{sub}} = \textit{sum of squares operator.}$$

A more general statement

Let G be a Lie group of dimension $\dim G = n$. Then on G we have two types of Haar measures. A

- left-invariant n-form μ_L, *(left-Haar measure)* i.e.

$$\int_G f(a * g)\mu_L(g) = \int_G f(g)\mu_L(g), \qquad \forall\, a \in G,\ \forall f \in L^1(G),$$

- right-invariant n-form μ_R, *(right-Haar measure)* i.e.

$$\int_G f(g * a)\mu_R(g) = \int_G f(g)\mu_R(g), \qquad \forall\, a \in G,\ \forall f \in L^1(G),$$

Definition

The group G is called unimodular if μ_L and μ_R are proportional.

Example: Let G be a nilpotent Lie group or $G = \mathrm{SL}(2)$ or $G = SO(3)$.

Proposition (A. Agrachev, U. Boscain, J.-P. Gauthier, F. Rossi, 2009)

Let $(G, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a left-invariant sub-Riemannian structure on a unimodular group G.

Then the intrinsic sub-Laplacian Δ_{sub} is a sum of squares of vector fields (i.e. it has no first order term).

Next Goal: *What are the analytic properties of Δ_{sub}? What can be said about the subelliptic heat flow?*

Let $(M, \mathcal{H}, \langle\cdot,\cdot\rangle)$ be a regular SR-manifold. Consider a local orthonormal frame for $\mathcal{H}$

$$[X_1, \cdots, X_m] \qquad \textit{with} \qquad m \leq n = \dim M.$$

Seen before: The intrinsic sub-Laplacian Δ_{sub} can be expressed in the form:

$$\Delta_{\mathrm{sub}} = \sum_{i=1}^{m} \Big[X_i^2 + \mathrm{div}_\omega(X_i)X_i\Big].$$

Hypoellipticity

Theorem (L. Hörmander, 1967)

Let $\Omega \subset \mathbb{R}^n$ be open. Consider C^∞- vector fields $[X_0, \cdots, X_m]$ with

$$\text{rank Lie}\Big[X_0, \cdots, X_m\Big] = n, \quad x \in \Omega \qquad \textit{(Hörmander condition)}.$$

The differential operator $\mathcal{L}$ is hypoelliptic:

$$\mathcal{L} := \sum_{j=1}^m X_j^2 + X_0 + c \qquad c \in C^\infty(\Omega)$$

Remarks

- An operator P is called hypoelliptic if

$$Pu = f \quad \textit{with} \quad f, u \in \mathcal{D}'(\Omega)$$

  implies: Let $\Omega_0 \overset{\text{open}}{\subset} \Omega$ and $f \in C^\infty(\Omega_0)$, then $u \in C^\infty(\Omega_0)$.
- The hypoellipticity statement in the Hörmander's Theorem follows via sub-elliptic estimates:

$$\|u\|_{s-\delta} \leq C_D\big(\|Au\|_s + \|u\|_0\big), \qquad u \in C_0^\infty(\underset{\substack{\uparrow \\ \text{bounded domain}}}{D})$$

- elliptic operators (e.g. the Laplace operator on a Riemannian manifold) are hypoelliptic (elliptic regularity).

Hörmander theorem: the version on manifolds

Theorem (L. Hörmander, 1967)

Let $\mathcal{L}$ be a differential operator on a manifold M, that locally in a neighborhood U of any point is written as

$$\mathcal{L} = \sum_{i=1}^{m} X_i^2 + X_0,$$

where $X_0, X_1, \cdots, X_m$ are C^∞ - vector fields with

$$\text{Lie}_q\Big\{X_0, X_1, \cdots, X_m\Big\} = T_qM \quad \forall \ q \in U.$$

Then $\mathcal{L}$ is hypoelliptic. In particular:

The intrinsic sub-Laplacian on a SR-manifold M is hypoelliptic.

Example: Kolmogorov operator

At the beginning of the 20th century:

A prototype of a kind of operator studied by A. N. Kolmogorov in relation with diffusion phenomena is the following:

Example: *Kolmogorov operator* **(proto-type)**

$$K = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{y_j} - \partial_t, \qquad mit \quad (x, y, t) \in \mathbb{R}^{2n+1}$$

"sum of squares + a first order term.

$x=$ *velocity* and $y:=$*position*.

Operator with *non-negative degenerate characteristic form.*

Heat kernel of the Sub-Laplacian

Definition

The heat kernel of the sub-Laplacian Δ_{sub}

$$K(t; x, y) : (0, \infty) \times M \times M \longrightarrow \mathbb{R}$$

is the fundamental solution of the heat operator:

$$P := \frac{\partial}{\partial t} - \Delta_{\text{sub}},$$

i.e. $K(t; x, y)$ fulfills

$$\begin{cases} PK(t; \cdot, y) = 0, & \text{for all } t > 0 \\ \lim_{t \downarrow 0} K(t; x, \cdot) = \delta_x, & \text{in the distributional sense.} \end{cases}$$

Applications: Hypoelliptic diffusion and human vision. Image reconstruction via non-isotropic diffusion. *(Boscain, Citti, Sarti,* $\cdots$

Remarks

We assume that M is complete as a metric space.

- Based on the essentially selfadjointness of Δ_{sub} on $C_c^\infty(M)$ the existence and uniqueness of the heat kernel is guaranteed. The details are discussed in a paper by R. Strichartz. [1]
- Hörmander's theorem also implies the hypoellipticity of the heat operator $P := \frac{\partial}{\partial t} - \Delta_{\text{sub}}$. Since the heat kernel solves

  $$PK(t; \cdot, y) = 0$$

  and is symmetric in the space variables, it follows that K is a smooth kernel on $\mathbb{R}_+ \times M \times M$.

[1] Robert S. Strichartz. Sub-Riemannian geometry. J. Differential Geom., 24(2):221 - 263, 1986.

From analysis to geometry and back

Intuition: Let $x, y \in M$ *(Riemannian manifold)*:

$$\textit{heat kernel} = K(t; x, y) = \text{"Heat flow from } x \text{ to } y \text{ at time } t\text{"}$$

"Meta-Theorem"

The heat kernel of the Sub-Laplacian Δ_{sub} has the form of a path integral:

$$K(t; x, y) = \int_{P_t(x,y)} e^{-S_t(\gamma)} d\mu_t(\gamma).$$

- $P_t(x, y)$ = space of curves, connecting x and y.
- $S_t(\gamma)$, classical action

$$S_t(\gamma) = \frac{1}{2} \int_0^1 \|\dot{\gamma}(s)\|^2 ds.$$

- μ_t, a "measure" on $P_t(x, y)$.

Example: heat asymptotic (ellipt. case)

Let $X_1, \cdots, X_n$ be vector fields on $\mathbb{R}^n$, linear independent at each point $x \in \mathbb{R}^n$. Consider

$$\Delta = -\frac{1}{2} \sum_{j=1}^{n} X_j^2 + (\text{lower order terms}).$$

Heat kernel

The heat kernel of Δ has the following asymptotic behaviour as $t \downarrow 0$:

$d(x, x')$ =Riemannian distance between x and x', with x near x'.

$$K(t; x, x') = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{d(x,x')^2}{2t}} \left[a_0(x, x') + a_1(x, x')t + \cdots \right].$$

Example: Heat trace asymptotic (ellip. case)

Let M be a compact Riemannian manifold with $\partial M \neq 0$

- Δ the Laplace-Beltrami operator *(zero Dirichlet boundary conditions)*.
- $\sigma(\Delta) = \{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ the spectrum (=eigenvalues) of Δ.

Then:

$$\underbrace{\sum_{j=1}^{\infty} e^{-t\lambda_j}}_{=\text{ heat trace}} \sim C_0 t^{-\frac{n}{2}} + C_1 t^{-\frac{n-1}{2}} + C_2 t^{-\frac{n-2}{2}} + \cdots, \qquad (t \downarrow 0).$$

With geometric quantities:

$$C_0 = \frac{\text{Vol}(M)}{(2\pi)^{\frac{n}{2}}}, \quad C_1 = \frac{\text{Vol}(\partial M)}{4(2\pi)^{\frac{n-1}{2}}}, \text{ and } C_2 = \frac{1}{6(2\pi)^{\frac{n-2}{2}}} \int_M \overset{\substack{\text{scalar curvature}\\ \downarrow}}{R(x)}dx\,.$$

New effects: SR-geometry

Geometry:

- end points of geodesics cannot be parametrized by initial velocities $(m < n)$.
- Even locally there may be finitely many (> 1) or infinitely many SR-geodesics between $x, y \in M$.
- There may be singular geodesics.

Analysis:

- Let $x_0 \in M$ be fixed. The map $x \mapsto d_{cc}^2(x_0, x)$ is not smooth.
- Even locally: the leading exponent in the asymptotic of the heat kernel $K(t, x, y)$ as $t \to 0$ depends on the position of x and y

Example: *SR-geodesic on the Heisenberg group* $\mathbb{H}_3$.

New effects: asymptotic behavior of the heat kernel

$K(t, x, y)$: heat kernel (HK) of the Sub-Laplacian:

Theorem, (Léandre 1987)

The following asymptotic hold:

$$\lim_{t \downarrow 0} t \log K(t; x, y) = -\frac{d_{sR}(x, y)^2}{2}.$$

(d_{sR} = Carnot-Carathéodory metric)

In case of a Riemannian manifold this relation is called *Varadhan formula*.

Heat asymptotic

The following improvements of the previous result are known:

Theorem, (Ben Arous 1989)

$$K(t; x, y) \sim t^{-\frac{\dim M}{2}} e^{-\frac{d_{CC}(x,y)^2}{2t}} \Big[a_0(x, y) + \mathcal{O}(\sqrt{t}) \Big], \qquad (t \downarrow 0)$$

if $x \neq y$ and x is not in the cut-locus of y.

Theorem, (Ben Arous, Léandre 1991)

Asymptotic on the diagonal

$$K(t; x, x) = \frac{C + O(\sqrt{t})}{t^{\frac{Q}{2}}} \quad \textit{with} \quad Q = \textit{Hausdorff dimension of } M.$$

Spectral zeta functions

Another point of view is the analysis of the spectral zeta function of the sub-Laplacian:

Spectral zeta function

Let A denote a non-negative operator with discrete spectrum

$$\sigma(A) = \{0 \leq \lambda_1 < \lambda_2 < \lambda_3 \cdots\},$$

where λ_j are eigenvalues of finite multiplicity $m(\lambda_j)$. The spectral zeta function of A is defined by:

$$\zeta_A(s) := \sum_{\lambda_j \neq 0} \frac{m(\lambda_j)}{\lambda_j^s}.$$

Question: *In particular, let $A = \Delta_{\text{sub}}$. What can be said about relations between geometric data and the meromorphic structure of $\zeta_{\Delta_{\text{sub}}}$ (meromorphic extension, pole distribution, residues, singularities in $s = 0$)?*

Next goal

"For a class of hypo-elliptic Hörmander operators generalizing the Kolmogorov operator study the small time heat kernel expansion [2] and a relation to a problem in control theory."

[2] D. Barilari, E. Paoli, *Curvature terms in small time heat kernel expansion for a model class of hypoelliptic Hörmander operators*, Nonlinear analysis 164 (2017), 118-134.