# Small time heat kernel expansion for a class of model hypoelliptic Hörmander operators

Winterschool in Geilo, Norway

Wolfram Bauer

Leibniz U. Hannover

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# Outline

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1. A class of hypoelliptic operators

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2. On a linear optimal control problem

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# Plan of the talk

We will study the on- and off-diagonal heat kernel expansion for a class of hypoelliptic operators that generalizes the Kolmogorov operator:

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We will study the on- and off-diagonal heat kernel expansion for a class of hypoelliptic operators that generalizes the Kolmogorov operator:

*Kolmogorov operator*

$$K = \sum_{j=1}^{n} \partial_{x_j}^2 + \sum_{j=1}^{n} x_j \partial_{y_j} - \partial_t, \quad \mathit{mit} \quad (x, y, t) \in \mathbb{R}^{2n+1}$$

*"sum of squares + a first order term.*

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*"sum of squares + a first order term*.

$x=$ *velocity* and $y$:=*position*.

By Hörmander's theorem this operator is hypoelliptic and admits a smooth heat kernel. We consider it as a model operator for the sub-Laplacian.

**The presentation is based on a work by D. Barilari and E. Paoli.** [1]

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# A class of hypoelliptic operators

Let $A = (a_{jh}) \in \mathbb{R}^{n\times n}$ and $B = (b_{il}) \in \mathbb{R}^{n\times k}$. Consider the second order differential operator

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\sum_{j,h=1}^{n}(BB^*)_{jh}\frac{\partial^2}{\partial x_j \partial x_h} + \sum_{j=1}^{n}(Ax)j\frac{\partial}{\partial x_j} \\ &= \frac{1}{2}\sum_{i=1}^{k}X_i^2 + X_0 = Ax\cdot\nabla + \frac{1}{2}\mathrm{div}(BB^*\nabla),\end{aligned}$$

where

$$X_i = \sum_{j=1}^{n} b_{ji}\frac{\partial}{\partial x_j} \quad \textit{and} \quad X_0 = \sum_{j,h=1}^{n} a_{jh}x_h\frac{\partial}{\partial x_j}, \qquad i = 1,\cdots,k.$$

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**Question:** *Under which condition is the operator $\mathcal{L}$ hypoelliptic?*

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## Lemma

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$$\operatorname{Lie}\Big\{\big[\operatorname{ad}X_0\big]^j(X_i) \,:\, i = 1, \cdots k, j \in \mathbb{N}\Big\}_x = T_x\mathbb{R}^n \qquad \forall\, x \in \mathbb{R}^n.$$

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*These imply hypoellipticity of* $\mathcal{L}$ *and the existence of a smooth heat kernel.*

# The Kolmogorov operator

## Example

Consider the case $A, B \in \mathbb{R}^{2n \times 2n}$, where

$$B = \sqrt{2}\begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}.$$

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The corresponding heat operator is the Kolmogorov operator:

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The rank condition is fulfilled with $m = 2$:

$$\mathrm{rank}\big[B, AB\big] = \sqrt{2}\begin{bmatrix} 0_n & I_n & 0_n & 0_n \\ 0_n & 0_n & 0_n & I_n \end{bmatrix} = 2n.$$

# An explicit form of the heat kernel

## Definition

The heat kernel of $\mathcal{L}$

$$p(t;x,y):(0,\infty)\times M\times M\longrightarrow\mathbb{R}$$

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$$\begin{cases} Pp(t;\cdot,y) = 0, & \text{for all } t > 0 \\ \lim_{t\downarrow 0} p(t;x,\cdot) = \delta_x, & \text{in the distributional sense.} \end{cases}$$

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**Goal:** *Study the asymptotic expansion of the kernel* $p(t;x,y)$ *as* $t\to 0$.

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## Theorem

*Let* $A \in \mathbb{R}^{n \times n}$ *and* $B \in \mathbb{R}^{n \times k}$ *with the rank condition*

$$\text{rank}\Big[B, AB, A^2B, \cdots, A^{m-1}B\Big] = n \quad \textit{for some } \; m \in \mathbb{N}.$$

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*Then:*

- *The heat operator* $\mathcal{L} - \frac{\partial}{\partial t}$ *is hypoelliptic and admits a smooth fundamental solution*

$$p(t;x,y) \in C^\infty\big(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n\big). \qquad (*)$$

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$$p(t;x,y) \in C^\infty\big(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n\big). \tag{$*$}$$

- *The kernel* $(*)$ *is explicitly known:*

$$p(t;x,y) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det D_t}} \exp\Big\{-\frac{1}{2}(y - e^{tA}x)^* D_t^{-1}(y - e^{tA}x)\Big\},$$

# An explicit form of the heat kernel

**Theorem (continued)**

Here $D_t \in \mathbb{R}^{n\times n}$ is the matrix:

$$D_t = e^{tA}\left(\int_0^t e^{-sA}BB^*e^{-sA^*}ds\right)e^{tA^*}.$$

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In particular, this matrix is invertible for all $t > 0$ (we will prove this later).

**Next goal:**

*From what is known in the elliptic set-up (next slide), one expect that, that the small time expansion of the heat kernel includes some geometric data and data on the drift term $X_0$.*

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**Q:** What happens in the sub-elliptic case?

# Laplace operator with drift term

## Theorem

*Let* $(M, g)$ *be a* *Riemannian manifold* *with* *Laplacian* $\Delta_g$ *and*

$$\mathcal{L} = \Delta_g + X_0.$$

# Laplace operator with drift term

**Theorem**

*Let* $(M, g)$ *be a Riemannian manifold with Laplacian* $\Delta_g$ *and*

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*Then the heat kernel of* $\mathcal{L}$ *has the following on-diagonal asymptotic expansion for small times:*

$$p(t; x_0, x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[ 1 - \left( \frac{\mathrm{div}(X_0)}{2} + \frac{\|X_0(x_0)\|^2}{2} - \frac{S(x_0)}{6} \right) t + O(t^2) \right],$$

*where* $S$ *denotes the scalar curvature of the Riemannian metric* $g$*.*

# Basics on optimal control problems

Let $\Omega \subset \mathbb{R}^k$ be a set and $f : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$.

With a given control function $\alpha : [0, \infty) \to \Omega$ and $\mathbf{x}_0 \in \mathbb{R}^n$ consider the system of ODE:

$$\begin{cases} \dot{\mathbf{x}}(t) = & f\big(\mathbf{x}(t), \alpha(t)\big), \qquad t > 0 \\ \mathbf{x}(0) = & \mathbf{x}_1. \end{cases} \tag{$*$}$$

We will call the solution the response of the system.

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## Definition

The set of admissible controls is

$$\mathcal{A} := \Big\{\alpha : [0, \infty) \to \Omega \ : \ \alpha \text{ measurable}\Big\}.$$

# Basics on optimal control

There is the question of controllability:

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## Controllability problem (special case)

Given an initial point $x_1 \in \mathbb{R}^n$ and an end point $x_2 \in \mathbb{R}^n$. Does there exist a control $\alpha(t)$ and a time $t_0 > 0$ with

$$\mathbf{x}(t_0) = x_2 \in \mathbb{R}^n,$$

where $\mathbf{x}(t)$ is a solution of the system $(*)$ of ODE's?

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For our later purpose it is sufficient to consider linear systems where we can answer the question:

Let $A = (a_{jh}) \in \mathbb{R}^{n \times n}$ and $B = (b_{il}) \in \mathbb{R}^{n \times k}$. With $T > 0$ consider:

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$$\begin{cases} \dot{\mathbf{x}} & = A\mathbf{x} + Bu \\ \mathbf{x}(0) & = x_1 \in \mathbb{R}^n, \end{cases} \quad \textit{where} \quad u \in L^\infty([0,T], \mathbb{R}^k). \qquad (**)$$

# A linear control problem

For a given control $u$ we write $\mathbf{x}_u : [0, T] \to \mathbb{R}^n$ for the solution of the initial value problem $(**)$. These are the admissible curves.

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**Solution:**

$$\mathbf{x}_u(t) = e^{tA}x_1 + e^{tA}\int_0^t e^{-sA}Bu(s)ds.$$

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(a) *A solution to the controllability problem for* $(**)$ *with end point* $x_2 \in \mathbb{R}^n$ *and time* $T > 0$ *exists.*

(b) *There is a control* $u \in L^\infty([0, T], \mathbb{R}^k)$ *such that*

$$x_2 = e^{TA}x_1 + e^{TA}\int_0^T e^{-sA}Bu(s)ds.$$

# Linear control problem

## Example

Consider the following special case: Let $(x_0, y_0)^t = 0$ and

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ u_2 \end{pmatrix} \text{ where } \quad A = 0 \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the $x$-component of a solution is constant end points $(x_1, y_1)$ with $x_1 \neq 0$ cannot be reached for any control $u$".

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Let $t > 0$ and consider the following two reachable sets:

$$\mathcal{C}(t) := \textit{initial points } x_0 \textit{ for which there is a control } u \textit{ such that } \mathbf{x}_u(t) = 0.$$

$$\mathcal{C} := \bigcup_{t>0} \mathcal{C}(t) = \textit{overall reachable set.}$$

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## Theorem (rank condition)

*The following statements are equivalent:* [a]

(i) rank $G(A,B) = n$,

(ii) $0 \in \overset{\circ}{\mathcal{C}}$ *(interior of $\mathcal{C}$).*

[a] J. Macki, A. Strauss, *introduction to optimal control*, Springer, 1982

# Optimal control problem

Now we add a cost functional to the controlled ODE. With $T > 0$ consider

$$\begin{cases} \dot{\mathbf{x}} & = A\mathbf{x} + B\mathbf{u}, \qquad \textit{where} \quad \mathbf{u} = (u_1, \cdots, u_k) \in L^\infty([0, T], \mathbb{R}^k) \\ J_T(u) & = \frac{1}{2}\int_0^T \sum_{i=1}^k |u_i(s)|^2 ds. \end{cases}$$

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**Problem**: *Among all solutions* $x_u := [0,T] \to \mathbb{R}^n$ *corresponding to the control* $\mathbf{u}$ *we want to minimize the cost* $J_T(u)$.

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Consider the value function:

$$S_T(x_1, x_2) = \inf \Big\{ J_T(u) \, : \, u \in L^\infty([0,T], \mathbb{R}^k), x_u(0) = x_1, x_u(T) = x_2 \Big\}.$$

This function is finite for all $T > 0$ and $x_1, x_2 \in \mathbb{R}^n$ by the rank condition.

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## Definition

A control $u$ that realizes the minimum is called an optimal control. The corresponding trajectory

$$x_u : [0,T] \to \mathbb{R}^n$$

is an optimal trajectory.

# Optimal control problem

**Q:** *How to find an optimal control?*

We assign to the optimal control problem an Hamiltonian, i.e. a function on the cotangent bundle:

$$H(x,p) := p^*Ax + \frac{1}{2}p^*BB^*p, \quad \textit{where} \quad (x,p) \in T^*\mathbb{R}^n \cong \mathbb{R}^{2n}.$$

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This induces a Hamilton system:

$$\begin{cases} \dot{p} &= -H_x = -A^* p \\ \dot{x} &= H_p = Ax + BB^* p. \end{cases} \tag{HS}$$

# Optimal control problem

**Q:** *How to find an optimal control?*

We assign to the optimal control problem an Hamiltonian, i.e. a function on the cotangent bundle:

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## Proposition

Optimal trajectories are projections $x(t)$ of the solution $(x(t), p(t))$ of (HS). The control realizing the optimal trajectory is uniquely given by:

$$u_{\text{op}}(t) = B^*p(t).$$

# Optimal control problem

Here is the explicit solution of the Hamilton system (HS) with initial condition $(x_1, p_1) \in T_{x_1}\mathbb{R}^n$:

$$\begin{cases} p(t) &= e^{-tA^*} p_1 \\ x(t) &= e^{tA}\Big(x_1 + \int_0^t e^{sA} B \underbrace{B^* e^{-sA^*} p_1}_{=u_{\text{op}}(s)} ds\Big). \end{cases} \qquad \text{(SHS)}$$

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For each $t > 0$ we define

$$\Gamma_t = \int_0^t e^{sA} B B^* e^{-sA^*} ds \in \mathbb{R}^{n \times n}.$$

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**Next step:** *We want to show that the matrix $\Gamma_t$ is invertible. Then we solve the last equation in (SHS) for $p_1$.*

Recall the controllability matrix:

$$G(A, B) = \underbrace{\Big[B, AB, A^2B, \cdots, A^{m-1}B\Big]}_{=n\times(m\cdot k)-\text{matrix}}.$$

# An invertibility condition

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**Lemma**

*Assume that* $\operatorname{rank} G(A,B) = n$*, then for all* $t > 0$ *the matrix-valued integral*

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*is invertible.*

**Proof:** Let $x \in \mathbb{R}^n$ such that $\Gamma_t x = 0$. Then

$$0 = \left\langle \int_0^t e^{-sA} BB^* e^{-sA^*} ds \cdot x, x \right\rangle = \int_0^t \left\| B^* e^{-sA^*} x \right\|^2 ds.$$

Therefore, we have $0 = B^* e^{-sA^*} x$ for all $s \in [0,t]$.

# Proof (continued)

Taking the transpose of the last equation, we find for all $s \in [0, t]$:

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In particular, we may choose $s = 0$. Then we find:

$$0 = x^* B = x^* AB = \cdots = x^* A^{m-1} B.$$

Since the controllability matrix

$$G(A, B) = [B, AB, A^2 B, \cdots A^{m-1} B]$$

has linear independent rows (maximal rank $n$) we conclude that $x = 0$. Hence $\Gamma_t$ is injective and therefore invertible.

# The value of the value function

**Next goal:** *Calculate the value function.*

Let us go back to the solution of the Hamilton system , which stands behind the optimal control problem:

$$\begin{cases} p(t) & = e^{-tA^*} p_1 \\ x(t) & = e^{tA}\Big(x_1 + \Gamma_t \cdot p_1\Big). \end{cases} \tag{SHS}$$

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Since $\Gamma_t$ is invertible for any $t > 0$ we can solve the 2nd equation for $p_1$ with $x_2 = x(T)$ [2]:

$$p_1 = \Gamma_T^{-1}\big(e^{-TA}x_2 - x_1\big) \qquad (T > 0).$$

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The optimal control is given by $u_{\mathrm{op}}(t) = B^* p(t)$ and therefore one can calculate:

$$S_T(x_1, x_2) = \inf\Big\{J_T(u) \;:\; u \in L^\infty([0,T], \mathbb{R}^k), x_u(0) = x_1, x_u(T) = x_2\Big\}.$$

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## Corollary

The value function $S_T(x_1, x_2)$ is smooth in $(T, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$.

# The geodesic cost

Let $x_1 \in \mathbb{R}^n$ be fixed and let $x_u(t)$ be an optimal trajectory of the problem:

$$\begin{cases} \dot{\mathbf{x}} & = A\mathbf{x} + B\mathbf{u}, \qquad \textit{where} \quad \mathbf{u} = (u_1, \cdots, u_k) \in L^\infty([0,T], \mathbb{R}^k) \\ J_T(u) & = \frac{1}{2}\int_0^T \sum_{i=1}^k |u_i(s)|^2 ds \end{cases}$$

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## Definition

The geodesic cost corresponding to $x_u$ is the family $\{c_t\}_t$ of functions:

$$c_t(x) = -S_t\big(x, x_u(t)\big), \qquad \textit{where} \quad x \in \mathbb{R}^n.$$

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We can calculate the geodesic cost explicitly from our previous formulas:

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Recall that $x_u(t)$ is the solution of the Hamilton system

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**Lemma**

*The geodesic cost is obtained by*

$$c_t(x) = -S_t\big(x, x_u(t)\big) = -\frac{1}{2} p_1^* \Gamma_t p_1 + p_1^*(x - x_1) - \frac{1}{2}(x - x_1)^* \Gamma_t^{-1}(x - x_1).$$

# Proof of the Lemma

**Proof**: We use our explicit formula for $S_t(x, x_u(t))$:

Let $v(s)$ be an optimal trajectory which connects $x$ and $x_u(t)$. Then

$$v(s) = e^{sA}\big(x + \Gamma_s \tilde{p}_1\big) \quad \textit{with some} \quad \tilde{p}_1 \in \mathbb{R}^n.$$

We use the condition $v(t) = x_u(t) = e^{tA}(x_1 + \Gamma_t p_1)$ to determine $\tilde{p}_1$:

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Insert this expression into our previous formula for the value function

$$\begin{aligned} c_t(x) = -S_t(x, x_u(t)) &= -\frac{1}{2}\tilde{p}_1^* \Gamma_t \tilde{p}_1 \\ &= \frac{1}{2}\Big(\Gamma_t^{-1}(x_1 - x) + p_1\Big)^* \Gamma_t \Big(\Gamma_t^{-1}(x_1 - x) + p_1\Big). \end{aligned}$$

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Combining terms give the result. □

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With the previous notation put:

$$\mathfrak{Q}(t) = B^* \left( d^2_{x_0} \dot{c}_t \right) B = -\frac{d}{dt} B^* \Gamma_t^{-1} B.$$

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  *"small time heat kernel expansion"*.

# Towards the coefficients in the heat kernel expansion

## Theorem (A. Agrachev, D. Barilari, L Rizzi)

*Let $x_u : [0, T] \to \mathbb{R}^n$ be an optimal trajectory of the optimal control problem and $\mathfrak{Q}(t)$ the corresponding family of quadratic forms:*

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(c) *There is a trace formula: with $k_i = \dim \operatorname{span} \{B, AB, \cdots, A^{i-1}B\}$:*

$$\operatorname{trace} \mathcal{I} = \sum_{i=1}^{m} (2i-1)\big(k_i - k_{i-1}\big).$$

# Main results

## Theorem (D. Barilari, E. Paoli, 2017)

*Let* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times k}$ *and* $x_0 \in \mathbb{R}^n$. *Consider the hypo-elliptic operator:*

$$\mathcal{L} = Ax \cdot \nabla + \frac{1}{2}\mathrm{div}(BB^*\nabla) \quad \textit{(with rank condition)}$$

*with heat kernel* $p(t;x,y) \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n)$. *Assume that* $Ax_0 = 0$.

$$p(t,x_0,x_0) = \frac{t^{-\frac{1}{2}\mathrm{tr}\mathcal{I}}}{(2\pi)^{\frac{n}{2}}\sqrt{c_0}}\left\{\sum_{i=0}^{\ell} a_i t^i + O(t^{\ell+1})\right\} \quad (t \to 0),$$

*where*

$$a_i = P_i\Big(\mathrm{tr}\,A, \mathrm{tr}\,\mathfrak{O}^{(0)}, \cdots, \mathrm{tr}\,\mathfrak{O}^{(i-2)}\Big), \quad \textit{and} \quad P_i = \text{polynomials}.$$

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*In particular:* $a_1 = -\frac{\mathrm{tr}\ A}{2}$ *and* $a_2 = \frac{(\mathrm{tr}\ A)^2}{8} + \frac{\mathrm{tr}\ \mathfrak{O}^{(0)}}{4}$.

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*With* $x_1, x_2 \in \mathbb{R}^n$ *consider the minimal cost function again:*

$$S_T(x_1, x_2) = \inf \left\{ J_T(u) \ : \ u \in L^\infty([0, T], \mathbb{R}^k), x_u(0) = x_1, x_u(T) = x_2 \right\}.$$

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*Then there is the following off-diagonal small time heat kernel asymptotic:*

$$p(t;x_1,x_2) \frac{t^{-\frac{1}{2}\mathrm{tr}\mathcal{I}}}{(2\pi)^{\frac{n}{2}}\sqrt{c_0}} e^{-S_t(x_1,x_2)} \left\{ \sum_{i=0}^{\ell} a_i t^i + O(t^{\ell+1}) \right\} \qquad (t \to 0).$$

# Main results

**Theorem (D. Barilari, E. Paoli, 2017)**

*With* $x_1, x_2 \in \mathbb{R}^n$ *consider the minimal cost function again:*

$$S_T(x_1, x_2) = \inf \Big\{ J_T(u) \;:\; u \in L^\infty([0, T], \mathbb{R}^k), x_u(0) = x_1, x_u(T) = x_2 \Big\}.$$

*Then there is the following off-diagonal small time heat kernel asymptotic:*

$$p(t; x_1, x_2) \frac{t^{-\frac{1}{2}\mathrm{tr}\mathcal{I}}}{(2\pi)^{\frac{n}{2}}\sqrt{c_0}} e^{-S_t(x_1,x_2)} \left\{ \sum_{i=0}^{\ell} a_i t^i + O(t^{\ell+1}) \right\} \qquad (t \to 0).$$

*The coefficients* $a_i$ *are the ones from the last theorem.*

# Main results

In the final result we consider the case $Ax_0 \neq 0$. With $i = 1, \cdots, m$ put:

$$E_i = \text{span}\Big\{ A^i Bx \ :\ x \in \mathbb{R}^k,\ 0 \leq j \leq i-1 \Big\} \subset \mathbb{R}^n.$$

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**Observation**

Now, the small time heat kernel expansion depends on the level $E_j$ in which we find $Ax_0$:

# Main results

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(i) *If $Ax_0 \in E_1$, then we have* polynomial decay *as $t \to 0$:*

$$p(t; x_0, x_0) = \frac{t^{-\frac{1}{2}\mathrm{tr}\mathcal{I}}}{(2\pi)^{\frac{n}{2}}\sqrt{c_0}} \left\{ 1 - \left( \frac{\mathrm{tr}\ A}{2} + \frac{|Ax_0|^2}{2} \right) t + O(t^2) \right\}.$$

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(ii) *If $Ax_0 \in E_i \setminus E_{i-1}$ for $i > 1$, then we have exponential decay to zero: There is $C > 0$ such that:*

$$p(t; x_0, x_0) = \frac{t^{-\frac{1}{2}\mathrm{tr}\mathcal{I}}}{(2\pi)^{\frac{n}{2}}\sqrt{c_0}} \exp\left\{ \frac{C + O(t)}{t^{2i-3}} \right\} \qquad (t \to 0).$$

**Remark**: The case (i) corresponds to the elliptic situation with zero scalar curvature.

# Laplace operator with drift term

Here is the formula again:

**Theorem**

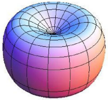
*Let* $(M, g)$ *be a Riemannian manifold with Laplacian* $\Delta_g$ *and*

$$\mathcal{L} = \Delta_g + X_0.$$

*Then the heat kernel of* $\mathcal{L}$ *has the following on-diagonal asymptotic small time-expansion:*

$$p(t; x_0, x_0) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \left[ 1 - \left( \frac{\mathrm{div}(X_0)}{2} + \frac{\|X_0(x_0)\|^2}{2} - \frac{S(x_0)}{6} \right) t + O(t^2) \right],$$

*where* $S$ *denotes the scalar curvature of the Riemannian metric* $g$.



**Thank you for your attention!**