Fundamental solution of degenerate operators

Winterschool in Geilo, Norway

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Leibniz U. Hannover

March 4-10. 2018

Outline

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1. Fundamental solutions and homogeneous vector fields

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2. A global lifting theorem by Folland

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3. Fundamental solutions via liftings

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Fundamental solution

Consider a linear PDO [1] P on $\mathbb{R}^n$ of order d, i.e.

$$P = \sum_{|\alpha| \leq d} a_\alpha(x) D_x^\alpha \quad \textit{where} \quad a_\alpha(x) \in C^\infty(\mathbb{R}^n, \mathbb{R})$$

where we use the standard notation

$$D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \quad \textit{for} \quad \alpha \in \mathbb{N}_0^n.$$

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Definition

Call $\Gamma : \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ x \neq y\} \to \mathbb{R}$ a global fundamental solution if

- For every $x \in \mathbb{R}^n$ we have $\Gamma(x, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^n)$.
- For every $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \Gamma(x,y) P^* \varphi(y) dy = -\varphi(x).$$

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- Existence of the fundamental solution is not always guaranteed. Showing existence (or non-existence) can be complicated.
- In general, fundamental solutions are not unique, e.g. one may add a P-harmonic function h i.e.

$$Ph = 0$$

  to a fundamental solution and gets another one.

From the heat kernel to the inverse

Let $\mathcal{L}$ be a linear partial differential operator on $\mathbb{R}^n_x$ (smooth coefficients).

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Assumption:

$\mathcal{H}$ admits a heat kernel $\{p_t(x,y)\}_{t>0}$ which is integrable with respect to t. In particular:

$$\mathcal{L}_y p_t(x,y) = \partial_t p_t(x,y).$$

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Observation: At least formally, a fundamental solution of $\mathcal{L}$ is given by

$$\Gamma(x,y) = \int_0^\infty p_t(x,y)dt$$

(can be made precise in many cases).

From the heat kernel to the inverse

In fact (formal calculation): Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

$$\begin{aligned}\int_{\mathbb{R}^n} \Gamma(x,y)\mathcal{L}^*\varphi(y)dy &= \int_{\mathbb{R}^n}\int_0^\infty p_t(x,y)dt\mathcal{L}^*\varphi(y)dy \\ &= \int_{\mathbb{R}^n}\int_0^\infty \mathcal{L}_y p_t(x,y)dt\varphi(y)dy \\ &= \int_{\mathbb{R}^n}\int_0^\infty \partial_t p_t(x,y)dt\varphi(y)dy \\ &= \int_{\mathbb{R}^n}\Big[p_t(x,y)\Big]_0^\infty \varphi(y)dy \\ &= -\lim_{t\to 0}\int_{\mathbb{R}^n} p_t(x,y)\varphi(y)dy = -\varphi(y).\end{aligned}$$

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&= -\lim_{t\to 0}\int_{\mathbb{R}^n} p_t(x,y)\varphi(y)dy = -\varphi(y).
\end{aligned}$$

Here we have assumed that $\lim_{t\to\infty} p_t(x,y) = 0$.

Reducing the dimension

Here is another example:

Consider the Laplace operator Δ_n on $\mathbb{R}^n$ with $n > 2$ and let $p \geq 1$.

$$\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \quad \textit{and} \quad \Delta_{n+p} = \Delta_n + \sum_{j=n+1}^{n+p} \frac{\partial^2}{\partial x_j^2}.$$

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Observation

We can consider Δ_{n+p} for $p \geq 1$ as a

"lifting of Δ_n",

i.e. it acts on functions only depending on the variable $x_1, \cdots, x_n$ as Δ_n.

Reducing dimension

Lemma

We obtain the fundamental solution of Δ_n from the fundamental solution of Δ_{n+p} via a "fiber integration".

$$c\left(\sqrt{x_1^2+\cdots+x_n^2}\right)^{2-n} =$$

$$= \int_{\mathbb{R}^p} \left(\sqrt{x_1^2+\cdots+x_n^2+t_1^2+\cdots+t_p^2}\right)^{2-n-p} dt_1\cdots dt_p = (*).$$

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Proof: The change of variables $t=\|x\|\tau$ with $\tau\in\mathbb{R}^p$ and $x\neq 0$ gives:

$$(*)=\|x\|^p\int_{\mathbb{R}^p}\left(\|x\|^2+\tau^2\|x\|^2\right)^{\frac{2-n-p}{2}}d\tau$$

$$=\|x\|^{2-n}\int_{\mathbb{R}^p}(1+\tau^2)^{\frac{2-n-p}{2}}d\tau.$$

Lifting

We describe recent work by **S. Biagi and A. Bonfiglioli**: [2]

Let P be a PDO with smooth coefficients.

Definition

We call a PDO P_{lift} on $\mathbb{R}^n \times \mathbb{R}^p$ a lifting of P if

[2] S. Biagi, A. Bonfiglioli, *The existence of a global fundamental solution for homogeneous Hörmander operators via a global lifting method*, Proc. London Math. Soc. (3), 114 (2017), 855-889.

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(a) P_{lift} has smooth coefficients depending on $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p$,

(b) For every $f \in C^\infty(\mathbb{R})$:

$$P_{\text{lift}}(f \circ \pi)(x, \xi) = (Pf)(x) \quad \textit{where} \quad (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p.$$

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Here $\pi : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is the projection to the x-coordinates:

$$\pi(x, y) = x.$$

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Example and equivalent formulation

Observation: P_{lift} is a lifting of P if and only if

$$P_{\text{lift}} = P + R \quad \textit{where} \quad R = \sum_{\beta \neq 0} r_{\alpha,\beta}(x,\xi) D_x^{\alpha} D_{\xi}^{\beta}.$$

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Example

Consider the Grushin operator on $\mathbb{R}^2$:

$$\mathcal{G} := \left(\partial_{x_1}\right)^2 + \left(x_1 \partial_{x_2}\right)^2.$$

A lifting of $\mathcal{G}$ to $\mathbb{R}^3$ is given by:

$$\widetilde{\mathcal{G}} = \left(\partial_{x_1}\right)^2 + \left(\partial_{\xi} + x_1 \partial_{x_2}\right)^2 = \mathcal{G} + \underbrace{\partial_{\xi}^2 + 2x_1 \partial_{x_2} \partial_{\xi}}_{R}.$$

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Remark: *$\widetilde{\mathcal{G}}$ is up to a change of variables the sub-Laplacian of the Heisenberg group. Its heat kernel is known do to the group structure.*

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- *If yes, can we obtain $\widetilde{\Gamma}$ by "via integration over some variables" in Γ?*
- *If we do not know Γ explicitly but if we have estimates on Γ. Can we use them to obtain estimates on $\widetilde{\Gamma}$?*
- *Why should be lift at all? Adding more variables should make live more complicated.*

Homogeneous group and dilations

Let $G = (\mathbb{R}^n, *)$ be a nilpotent Lie group.

Definition (homogeneous group)

We call G a homogeneous group, it there is $\sigma = (\sigma_1, \cdots, \sigma_n) \in \mathbb{R}^n$ with

$$1 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$$

such that the dilation $\delta_\lambda : G \to G$ with

$$\delta_\lambda(x_1, \cdots, x_n) = \left(\lambda^{\sigma_1} x_1, \cdots, \lambda^{\sigma_n} x_n\right)$$

is an automorphism of G for every $\lambda > 0$, i.e.

$$\delta_\lambda(g) * \delta_\lambda(h) = \delta_\lambda\big(g * h\big).$$

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Remark:

The dilations $\{\delta_\lambda\}_\lambda$ form a one-parameter group of automorphisms.

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Example: Consider again the Heisenberg group $\mathbb{H}_3 \cong \mathbb{R}^3$ with dilation:

$$\delta_\lambda(x_1, x_2, x_3) := (\lambda x_1, \lambda x_2, \lambda^2 x_3) \quad \textit{with} \quad (\lambda > 0).$$

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Then we have

$$\begin{aligned}
&\delta_\lambda(x_1, x_2, x_3) \, * \, \delta_\lambda(y_1, y_2, y_3) \\
&\quad = \big(\lambda x_1, \lambda x_2, \lambda^2 x_3\big) \, * \, \big(\lambda y_1, \lambda y_2, \lambda^2 y_3\big) \\
&\quad = \Big(\lambda(x_1 + y_1), \lambda(x_2 + y_2), \underbrace{\lambda^2(x_3 + y_3) + \frac{1}{2}\big[\lambda x_1 \lambda y_2 - \lambda y_1 \lambda x_2\big]}_{=\lambda^2\left((x_3+y_3)+\frac{1}{2}\left[x_1 y_2 - y_1 x_2\right]\right)}\Big) \\
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&\quad = \delta_\lambda\Big((x_1, x_2, x_3) * (y_1, y_2, y_3)\Big).
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Therefore $\mathbb{H}_3$ is a homogeneous Lie group with $\sigma = (1, 1, 2)$.

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Definition

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$$X_i\big(f \circ \delta_\lambda\big) = \lambda\big(X_i f\big) \circ \delta_\lambda \quad \forall\, \lambda > 0, \ \forall\, f \in C^\infty(\mathbb{R}^n, \mathbb{R}).$$

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- fulfill the Hörmander bracket generating condition, i.e for all $g \in G$:

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Note: *If X and Y are δ_λ-homogeneous of degree d_1 and d_2, respectively. Then $[X, Y]$ is δ_λ-homogeneous of degree $d_1 + d_2$.*

Example:

Consider again the Grushin operator $\mathcal{G}$ on $\mathbb{R}^2$ defined by

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Observation

- The bracket generating condition is fulfilled with $m = 2$, since

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- X_1 and X_2 are homogeneous of degree 1: Let $g = (x_1, x_2)$, then

$$X_1\left(f \circ \delta_\lambda\right)(g) = \partial_{x_1}[f(\lambda x_1, \lambda x_2)] = \lambda(\partial_{x_1} f)(\lambda x_1, \lambda^2 x_2) = \lambda(X_1 f) \circ \delta_\lambda(g),$$
$$X_2\left(f \circ \delta_\lambda\right)(g) = x_1\partial_{x_2}[f(\lambda x_1, \lambda^2 x_2)] = \lambda(\lambda x_1)[\partial_{x_2} f] \circ \delta_\lambda(g) = \lambda(X_2 f) \circ \delta_\lambda(g).$$

From homogeneous vector fields to a nilpotent Lie algebra

Let $\mathcal{X} = \{X_1, \cdots, X_m\}$ be homogeneous vector fields on $\mathbb{R}^n$ with the previous assumptions. We consider the Lie algebra generated by $\mathcal{X}$:

$$\begin{aligned}\mathfrak{a} :&= \text{Lie}\Big\{X_1, \cdots, X_m\Big\} \\ &= \textit{smallest Lie subalgebra of vector fields on } \mathbb{R}^n \textit{ containing } \mathcal{X}.\end{aligned}$$

The δ_λ-homogeneity of the vector fields implies the following:

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Lemma

The Lie algebra $\mathfrak{a}$ is finite dimensional and it corresponds to a Carnot group

$$\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \oplus \cdots \oplus \mathfrak{a}_r \quad \textit{and} \quad \begin{cases} \big[\mathfrak{a}_1, \mathfrak{a}_{i-1}\big] = \mathfrak{a}_i, & 2 \leq i \leq r, \\ \big[\mathfrak{a}_1, \mathfrak{a}_r\big] = \{0\}. \end{cases}$$

From homogeneous vector fields to a nilpotent Lie algebra

Let $\mathcal{X} = \{X_1, \cdots, X_m\}$ be homogeneous vector fields on $\mathbb{R}^n$ with the previous assumptions. We consider the Lie algebra generated by $\mathcal{X}$:

$$\begin{aligned} \mathfrak{a} := \mathrm{Lie}\Big\{X_1, \cdots, X_m\Big\} \\ = \textit{smallest Lie subalgebra of vector fields on } \mathbb{R}^n \textit{ containing } \mathcal{X}. \end{aligned}$$

The δ_λ-homogeneity of the vector fields implies the following:

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Here, the "first level" is $\mathfrak{a}_1 = \mathrm{span}\{X_1, \cdots, X_m\}$.

From the nilpotent Lie algebra to a Carnot group

Reminder:

We can equip $\mathfrak{a}$ with a group structure via exponential coordinates: Via the Campbell-Baker-Hausdorff formula the product is:

$$X \diamond Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}\big[X, [X, Y]\big] - \frac{1}{12}\big[Y, [X, Y]\big] + \cdots \text{(finite)}.$$

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Summing up:

Lemma

Let $N := \dim \mathfrak{a}$. Then $G = (\mathfrak{a} \cong \mathbb{R}^N, \diamond)$ is a Carnot group with Lie algebra (isomorphic to) $\mathfrak{a}$. Moreover,

$$\mathfrak{a} = \mathrm{Lie}\Big\{X_1, \cdots, X_m\Big\}$$

is a Lie algebra of smooth vector fields on $\mathbb{R}^n$ (we can "exponentiate").

A family of dilations on $\mathfrak{a} \cong G$

Recall that $\mathfrak{a}$ has a stratification

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Definition

For each $\lambda > 0$ define a dilation $\{\delta_\lambda^{\mathfrak{a}}\}_\lambda$ on $\mathfrak{a} \cong \mathbb{R}^N$ via the decomposition of elements:

$$\delta_\lambda^{\mathfrak{a}}(X) = \sum_{k=1}^{r} \lambda^k a_k \quad \textit{where} \quad X = \sum_{k=1}^{r} a_k \quad \textit{and} \quad a_k \in \mathfrak{a}_k.$$

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Proof: It is sufficient to show that $\delta_\lambda^{\mathfrak{a}}$ induces a Lie algebra automorphism:

$$[\delta_\lambda^{\mathfrak{a}}(X), \delta_\lambda^{\mathfrak{a}}(Y)] = \Big[\sum_{j=1}^{r} \lambda^j a_j, \sum_{\ell=1}^{k} \lambda^\ell b_\ell\Big] = \sum_{j,\ell=1}^{r} \lambda^{j+\ell} \underbrace{[a_j, a_\ell]}_{\in \mathfrak{a}_{j+\ell}} = \delta_\lambda^{\mathfrak{a}}([X, Y]).$$

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We call the basis

$$\left[X_1, \cdots, X_m, X_1^{(2)}, \cdots X_{\ell_2}^{(2)}, \cdots, X_1^{(r)}, \cdots, X_{\ell_r}^{(r)}\right]$$

an adapted basis of the Lie algebra $\mathfrak{a}$. This basis gives the concrete identification between $\mathfrak{a}$ and $\mathbb{R}^N$.

Sub-Riemannian structure on the Carnot group

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Identifications

Via the above basis we make the following identifications:

Carnot group: $G \cong \mathfrak{a} \quad \longleftrightarrow \quad \mathbb{R}^N$,

dilation on G: $\delta_\lambda^{\mathfrak{a}} \quad \longleftrightarrow \quad D_\lambda(a) = \big(\lambda^{s_1} a_1, \cdots, \lambda^{s_N} a_N\big), \quad a \in \mathbb{R}^N.$

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The exponents s_j in the dilation are given by:

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Identify $[X_1, \cdots, X_m]$ with left-invariant vector fields $[J_1, \cdots, J_m]$ on the homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$. Then

$$\mathcal{H} = \text{span}\Big\{J_1, \cdots, J_m\Big\} \subset T\mathbb{R}^N$$

is a bracket generating distribution in the tangent bundle of $\mathbb{R}^N$.

Sub-Laplacian on $\mathbb{R}^N$

Observation

The homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$ is equipped via $\mathcal{H}$ with a Sub-Riemannian structure. Its (intrinsic) Sub-Laplacian has the form:

$$\Delta_{\text{sub},G} = J_1^2 + \cdots + J_m^2,$$

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Question: Now we have constructed two "sum-of-squares operators":

$$\mathcal{L} = X_1^2 + \cdots + X_m^2 \quad (\text{on } \mathbb{R}^n)$$
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- What is the relation between these operators?
- Can we use knowledge on $\Delta_{\text{sub},G}$ to study $\mathcal{L}$?

From the Carnot group back to $\mathbb{R}^n$

Let $X \in \mathfrak{a} = \text{Lie}\{X_1, \cdots, X_m\}$ be a δ_λ-homogeneous vector field on $\mathbb{R}^n$. Consider the induced integral curve starting in $0 \in \mathbb{R}^n$:

$$\Psi_t^X : \mathbb{R} \to \mathbb{R}^n \quad \textit{with} \quad \begin{cases} \frac{d}{dt}\Psi_t^X = X \circ \Psi_t^X, & t \in \mathbb{R} \\ \Psi_0^X = 0. \end{cases} \tag{$*$}$$

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Based on the δ_λ-homogeneity one can show that all vector fields $X \in \mathfrak{a}$ are complete, i.e. the induced flow $(*)$ exists for all times $t \in \mathbb{R}$.

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Consider the following map:

$$\pi : \mathbb{R}^N \to \mathbb{R}^n, \qquad \pi(a) = \Big(\Psi_t^{X_a}(0)\Big)_{|t=1},$$

where $\mathfrak{a} \ni X_a \longleftrightarrow a \in \mathbb{R}^N$ in our identification above.

Lifting theorem by Folland

Theorem (Folland, 1977)

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- π *is a polynomial map.*
- *If* $J_1, \cdots, J_N$ *are the left-invariant vector fields which correspond to the adapted basis of* $\mathfrak{a} \cong \mathbb{R}^N$*, then*

$$d\pi(J_i)(a) = X_i\big(\pi(a)\big), \quad \forall\, a \in \mathbb{R}^N,$$

where X_i *is in the adapted basis of* $\mathfrak{a}$*.*

[a] G.B. Folland, *on the Rothschild-Stein lifting theorem*, Comm. Partial Differential Equations 2 (1977), 161-207.

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Next:

Can one choose coordinates in Folland's lifting theorem in such a way that

$$\pi : \mathbb{R}^N \to \mathbb{R}^n$$

becomes just the projection onto the first n coordinates of a vector in $\mathbb{R}^N$.

Lifting sums of squares

Theorem (S. Biagi, A. Bonfiglioli, 2017)

Let $X_1, \cdots, X_m$ be δ_λ-homogeneous of degree 1 vector fields on $\mathbb{R}^n$ with

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Remark: One can construct the lifting explicitly!

Lifting sums of squares

With the above notation we have:

Theorem (S. Biagi, A. Bonfiglioli, 2017)

The sub-Laplacian

$$\Delta_{\mathrm{sub},G} = Z_1^2 + \cdots + Z_m^2$$

on the homogeneous Carnot group $(\mathbb{R}^N, \diamond, D_\lambda)$*, of the previous theorem is a lifting of the sum-of-squares operator:*

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Assumptions:

Let $X_1, \cdots, X_m$ be linearly independent smooth vector fields on $\mathbb{R}^n$ with:

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3. Define the sum-of-squares operator: $\mathcal{L} = \sum_{j=1}^m X_j^2$.
4. $G = (\mathbb{R}^N, \diamond, D_\lambda) =$ homogeneous Carnot group constructed above with sub-Laplacian:

$$\Delta_{\mathrm{sub}, G} = Z_1^2 + \cdots + Z_m^2.$$

A theorem by Folland

Homogeneous norm: $(\sigma_1, \cdots, \sigma_n, \sigma_1^*, \cdots, \sigma_p^*)$ hom. dimensions of D_λ:

$$h(x,\xi) = \sum_{j=1}^{n} |x_j|^{\frac{1}{\sigma_j}} + \sum_{k=1}^{p} |\xi_k|^{\frac{1}{\sigma_k^*}} \quad \textit{where} \quad (x,\xi) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p.$$

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(a) $\gamma_G \in C^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{R})$ *and* $\gamma_G > 0$ *on* $\mathbb{R}^N \setminus \{0\}$,

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Homogeneous norm: $(\sigma_1, \cdots, \sigma_n, \sigma_1^*, \cdots, \sigma_p^*)$ hom. dimensions of D_λ:

$$h(x,\xi) = \sum_{j=1}^{n} |x_j|^{\frac{1}{\sigma_j}} + \sum_{k=1}^{p} |\xi_k|^{\frac{1}{\sigma_k^*}} \quad \textit{where} \quad (x,\xi) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p.$$

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(d) *There is* $C > 0$ *with:* $C^{-1} h^{2-Q}(x,\xi) \leq \gamma_G(x,\xi) \leq C h^{2-Q}(x,\xi)$.

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$$\Gamma(x,y) := \int_{\mathbb{R}^p} \Gamma_G\big(x,0;y,\eta\big)\,d\eta \qquad (x \neq y)$$

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(b) *There is a global estimate:*

$$C^{-1}\int_{\mathbb{R}^p} h^{2-Q}\big((x,0)^{-1} \diamond (y,\eta)\big)\,d\eta \leq \Gamma(x,y) \leq$$

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With the previous notations: $\Gamma(x, y)$ *has the* δ_λ*-homogeneity:*

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Example

Consider the Grushin operator on $\mathbb{R}^2$ with dilation $\delta_\lambda(x_1, x_2) = (\lambda x_1, \lambda^2 x_2)$:

$$\mathcal{L} = X_1^2 + X_2^2 \quad \textit{where} \quad X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2}.$$

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- Liftings of $X_1 \to Z_1 = \partial_{x_1}$ and $X_2 \to Z_2 = x_1 \partial_{x_2} + \partial_\xi$ and

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Conclusion

The fundamental solution of $\mathcal{L}$ is given by the fiber integral:

$$\Gamma(x_1, x_2; y_1, y_2) =$$
$$= c \int_{\mathbb{R}} \frac{d\eta}{\sqrt{((x_1 - y_1)^2 + \eta^2)^2 + 4(2x_2 - 2y_2 + \eta(x_1 + y_1))^2}}.$$

Higher step groups and Grushin type operators

Example: Consider the Engel group $\mathcal{E}_4$ as a matrix group

$$\mathcal{E}_4 = \left\{ \begin{pmatrix} 1 & x & \frac{x^2}{2} & z \\ 0 & 1 & x & w \\ 0 & 0 & 1 & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \; : \; x, y, w, z \in \mathbb{R} \right\} \subset \mathbb{R}^{4\times 4}.$$

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A 3-step Carnot group

The Engel group $\mathcal{E}_4$ is the lowest dimensional Carnot group of step 3.

Calculate the left-invariant vector fields X and Y on $\mathcal{E}_4$ [3].

$$X := \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial w} + \left(\frac{w}{2} - \frac{xy}{12}\right)\frac{\partial}{\partial z},$$
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Lemma

The vector fields X and Y are skew-symmetric on $\mathcal{E}_4$. They span a bracket generating distribution:

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Consider the sub-group

$$\mathcal{N} = \{sX + tW \ : \ s, t \in \mathbb{R}\} \cong \mathbb{R}^2$$

of $\mathcal{E}_4 \cong \mathfrak{e}_4$. One obtains a fiber bundle

$$\rho : \mathcal{E}_4 \longrightarrow \mathcal{N}\backslash\mathcal{E}_4 \cong \mathbb{R}^2, \quad \text{where} \quad \rho(x, y, w, z) = \left(x, z + \frac{xw}{2} + \frac{yx^2}{6}\right).$$

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Observation

The vector fields X and Y descend via $d\rho$ to $\mathcal{N} \backslash \mathcal{E}_4$. We obtain the Grushin type operator

$$\mathcal{G} = -d\rho(X)^2 - d\rho(Y)^2 = -\frac{\partial^2}{\partial u^2} - \frac{u^4}{4} \frac{\partial^2}{\partial v^2}.$$

Perform a partial Fourier transform with respect to the variable v. We obtain a family of operators on $\mathbb{R}$

$$\mathcal{L}_\eta := -\frac{\partial^2}{\partial u^2} - \frac{u^4}{4}\eta^2 = \text{"quartic oscillator" if } \eta \neq 0.$$

These operators are elliptic if $\eta \neq 0$.

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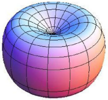
$$K^{\mathcal{G}}(t, \rho(x), y) = \int_{\mathbb{R}^2} K^{\Delta_{\text{sub}}^{\mathcal{E}_4}}(t, x, \Phi(u, y))\, du.$$

[a] trivialization means: $\rho \circ \Phi(x, y) = x$.



Thank you for your attention!