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$(4,7)$-decomposable solutions of 11-dimensional supergravity

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Plan of the talk:

- Introduction in supergravity + small history
- Main problem: Lorentzian mnfd $(\widetilde{M}^{3,1}, \widetilde{g})$, Riemannian mnfd (M^7, g)
- Examination of 11-dimensional (bosonic) supergravity equations on the product

$$(\mathscr{M}^{10,1} = \widetilde{M}^{3,1} \times M^7, g_{\mathscr{M}} = \widetilde{g} + g), \quad \text{Flux form: } \mathscr{F}^4 = f \cdot \mathrm{vol}_{\widetilde{M}} + F^4.$$

$\Rightarrow$ Analyse the system of equations $\Rightarrow$ Reduction of the problem to an examination of special 3-forms ϕ on (M^7, g) $\Rightarrow$ Relation with special G-structures:

- When ϕ is generic, or stable à la *Hitchin* $\Rightarrow$ Existence of solutions $\Longleftrightarrow$
  - $(\widetilde{M}^{3,1}, \tilde{g})$ is **Einstein** with negative Einstein constant
  - (M^7, g) is a **weak**-G_2 **manifold**

(Open) Question: What is the case when ϕ is not generic?

- Methodology: $\Rightarrow$ Relies on the homogeneous setting - Examples

$\rightsquigarrow$ Discussion of further *open problems*

11D supergravity equations (M theory)

A) Generalities

Supergravity theories are supersymmetric generalizations of General Relativity in various dimensions.

A supergravity action S consists of *bosonic* (ϕ_i) and *fermionic* (ψ_i) fields;

$$S = \text{bosonic part } (\phi_i, \nabla\phi_i, \ldots) + \text{fermionic part } (\psi_i, \nabla\psi_i, \ldots)$$

• Bosonic part: corresponds to gravitational and gauge field degrees of freedom (graviton (metric), p-form fields, etc.)

• Fermionic part: consists of matter degrees of freedom (gravitino (spin-3 Rarita-Schwinger field), gaugino, etc).

$\rightsquigarrow$ *Supersymmetry transformations* relate the bosonic and fermionic fields to each other.

$\Rightarrow$ Considering the fermionic fields to be **zero**, we obtain the *bosonic supergravity*, whose solutions give the consistent *geometric backgrounds* of the theory.

B) 11-dimensional supergravity

$D = 11$ bosonic supergravity action:

- 3-form A (potential)
- 4-form $\mathscr{F} = \mathrm{d}\,A$ (flux form)

$$S = \underbrace{\frac{1}{2}\int R \;\mathrm{d}\,\mathrm{vol}}_{\text{gravity action}} - \underbrace{\frac{1}{2}\int \mathscr{F} \wedge *\mathscr{F}}_{\text{Maxwell-like term}} - \underbrace{\frac{1}{6}\int A \wedge \mathscr{F} \wedge \mathscr{F}}_{\text{Chern-Simons term}}$$

$* : \Lambda^p(\mathscr{M}) \to \Lambda^{11-p}(\mathscr{M})$ star Hodge operator
$\Rightarrow$ The (bosonic) 11-dimensional supergravity background consists of an 11-dimensional Lorentzian (spin) manifold $(\mathscr{M}^{10,1}, g_{\mathscr{M}})$ with a 4-form $\mathscr{F}$, satisfying the following *field equations*:

Closure	$\mathrm{d}\,\mathscr{F}$	$=$	0
Maxwell	$\mathrm{d}*\mathscr{F}$	$=$	$(1/2)\mathscr{F} \wedge \mathscr{F}$
Einstein	$\mathrm{Ric}^{g_{\mathscr{M}}}(X, Y)$	$=$	$(1/2)\langle X \lrcorner \mathscr{F}, Y \lrcorner \mathscr{F}\rangle_{\mathscr{M}} - (1/6) g_{\mathscr{M}}(X, Y)\|\mathscr{F}\|^2_{\mathscr{M}}$

where

$$\langle X \lrcorner \mathscr{F}, Y \lrcorner \mathscr{F}\rangle_{\mathscr{M}} = \frac{1}{3!} g_{\mathscr{M}}(X \lrcorner \mathscr{F}, Y \lrcorner \mathscr{F}), \qquad \|\mathscr{F}\|^2_{\mathscr{M}} = \frac{1}{4!} g_{\mathscr{M}}(\mathscr{F}, \mathscr{F}).$$

Examples of 11D supergravity backgrounds

Set $\mathscr{M}^{10,1} := \widetilde{M}^{3,1} \times M^7$, or $\mathscr{M}^{10,1} := \widetilde{M}^{4,1} \times M^6$
In the presence of fluxes, $\widetilde{M}^{3,1}$ or $\widetilde{M}^{4,1}$ can be some $\mathscr{A}ntide\mathscr{S}itter$ space. – Examples:

$$\mathscr{F}reundin - \mathscr{R}udin: \quad \mathscr{M}^{10,1} = \underbrace{\widetilde{M}^{3,1}}_{\mathrm{Ad\,S}_4} \times \underbrace{M^7}_{\mathrm{S}^7} \quad \text{or} \quad \mathscr{M}^{10,1} = \underbrace{\widetilde{M}^{4,1}}_{\mathrm{Ad\,S}_5} \times \underbrace{M^6}_{\mathrm{S}^6}$$

$\rightsquigarrow$ One can consider also other types, e.g. $\mathrm{Ad\,S}^7 \times M^4$, $\mathrm{Ad\,S}^6 \times M^5$, etc.

In general, the appearing geometric structure **depends on the flux form!** – Examples:

- if $\mathscr{M}^{10,1} := \widetilde{M}^{3,1} \times M^7$ and $\mathscr{F} = 0 \implies M^7$ is $\mathscr{R}icci - flat \Rightarrow \mathrm{G}_2$-holonomy.
- if $\mathscr{M}^{10,1} := \widetilde{M}^{2,1} \times M^8$ and $\mathscr{F} = 0 \implies M^8$ is $\mathscr{R}icci - flat \Rightarrow \mathrm{Spin}_7$-holonomy.

SUGRA equations for $(4,7)$ decomposable case

- **Setting:** Oriented Lorenztian mnfd $(\mathscr{M}^{10,1} = \widetilde{M}^{3,1} \times M^7, g_{\mathscr{M}} = \tilde{g} + g,\ \mathscr{F} := \widetilde{F} + F)$, for some $\widetilde{F} \in \Lambda^4(\widetilde{M}^{3,1})$ and $F \in \Lambda^4(M^7)$.

- *Flux form*:

$$\mathscr{F} := \lambda \cdot \widetilde{\mathrm{vol}} + F\,, \quad \lambda \in \mathbb{R}.$$

Proposition. The closure condition and the Maxwell equation are simultaneously satisfied, iff

$$\mathrm{d}\, F = 0, \quad \text{and} \quad \mathrm{d} *_7 F = \lambda \cdot F.$$

If $\lambda = 0$, then the same equations are simultaneously satisfied iff $\mathrm{d}\, F = \mathrm{d} *_7 F = 0$.

Set $\phi := *_7 F \ \rightsquigarrow\ *_7\phi = F$.

Corollary. The Maxwell equation for the 4-form $\mathscr{F}$ is equivalent to the equation

$$\mathrm{d}\,\phi = \lambda *_7 \phi, \quad \text{where } \phi := *_7 F.$$

Moreover, the closure condition is equivalent to the relation $\mathrm{d} *_7\phi = 0$.

Definition. A 3-form $\phi \in \Omega^3(M)$ on a Riemannian 7-manifold (M, g) is called *special* if it is co-closed $(\mathrm{d} *_7\phi = 0)$ and satisfies the relation $\mathrm{d}\,\phi = f *_7 \phi$ for some constant $f \in \mathbb{R}$.

The $D = 11$ SUGRA equations for this specific Ansatz

- For this setting and in terms of the 3-form $\phi := *_7 F$ we obtain that

$$
\begin{aligned}
\mathrm{d} *_7 \phi &= 0 && (\mathscr{Closure}) \\
\mathrm{d}\,\phi &= \lambda *_7 \phi\,, \quad \lambda \in \mathbb{R} && (\mathscr{Maxwell}) \\
\mathrm{Ric}^{\tilde{g}}(\tilde{X}, \tilde{Y}) &= -\tfrac{1}{6}\,(2\lambda^2 + \|\phi\|^2)\,\tilde{g}(\tilde{X}, \tilde{Y}) && (\mathscr{Einstein}_1) \\
\mathrm{Ric}^{g}(X, Y) &= \tfrac{1}{6}\,(f^2 + 2\|\phi\|^2_M)\,g(X, Y) + q_\phi(X, Y) && (\mathscr{Einstein}_2)
\end{aligned}
$$

where q_ϕ is the symmetric bilinear form

$$q_\phi(X, Y) := -\frac{1}{2}\langle X \lrcorner \phi, Y \lrcorner \phi \rangle_M.$$

Proposition. Let $(\widetilde{M}, \tilde{g}, \widetilde{F} = \lambda \cdot \mathrm{vol}_{\widetilde{M}})$ be the 4-dimensional Lorentzian manifold of an eleven-dimensional supergravity background of the form $(\mathscr{M} = \widetilde{M} \times M, g_{\mathscr{M}} = \tilde{g} + g)$, with the flux 4-form $\mathscr{F} = \lambda \cdot \widetilde{\mathrm{vol}} + F$, with $f \in \mathbb{R}$. Then, $(\widetilde{M}, \tilde{g})$ is Einstein with negative Einstein constant $\Lambda := -\frac{1}{6}\,(2\lambda^2 + \|\phi\|^2)$.

Special gravitational Einstein 7-manifolds

Definition. A Riemannian 7-manifold (M^7, g, ϕ) with a special 3-form ϕ is said to be a **special gravitational Einstein manifold** if the pair (g, ϕ) is a solution of the supergravity Einstein equation:

$$\mathrm{Ric}^g(X,Y) = \frac{1}{6}\left(f^2 + 2\|\phi\|_M^2\right) g(X,Y) + q_\phi(X,Y)$$

- Note that a special gravitational Einstein 7-manifold is *not* necessarily an *Einstein manifold*
- The last equation is an extension of the Einstein equation by a stress-energy tensor associated to the 3-form ϕ

Theorem. Any $(4,7)$-decomposable solution $(\mathscr{M}^{10,1}, g_{\mathscr{M}}, \mathscr{F})$ of eleven-dimensional supergravity with flux 4-form

$$\mathscr{F} := \lambda \cdot \mathrm{vol}_{\widetilde{M}} + F^4\,, \quad F^4 := \star_7 \phi \in \Omega^4_{\mathrm{cl}}(M^7)\,, \quad \lambda \in \mathbb{R},$$

is a product of Lorentzian Einstein 4-manifold $(\widetilde{M}^{3,1}, \tilde{g})$ with negative Einstein constant and a gravitational special Einstein 7-manifold (M^7, g) with special 3-form $\phi \in \Omega^3(M^7)$.

Three basic classes of Riemannian 7-manifolds with a special 3-form ϕ

We consider three classes of special 3-forms on a Riemannian 7-manifolds and discuss the problem of solutions of supergravity Einstein equation for such manifolds.

A) Zero form $\phi = F = 0$.

B) Non zero harmonic form $\phi \neq 0, \lambda = 0$.

C) Non harmonic form $\phi \neq 0,\ \lambda \neq 0$.

(Note that in symmetric case any invariant form is parallel and case C) is impossible)

• *Case A :*

Proposition. The SUGRA Einstein equation for special 3-forms of Type A, reduces to the standard Einstein equation, i.e.

$$\mathrm{Ric}^g = (\lambda^2/6)g.$$

Consequently, using the flux 4-form $\mathscr{F} = \lambda \cdot \mathrm{vol}_{\widetilde{M}}$ we obtain a $(4,7)$-decomposable supergravity background, given by a product of a Lorentzian Einstein 4-manifold $(\widetilde{M}^{3,1}, \tilde{g})$ with Einstein constant $-\lambda^2/3$, and a Riemannian Einstein 7-manifold (M^7, g) with Einstein constant $\lambda^2/6$.

• *Case B :*
Proposition. The SUGRA Einstein equation for a special 3-form $\phi \neq 0$ on M^7 of Type B, reduces to the equation

$$\mathrm{Ric}^g = \frac{1}{3}\|\phi\|_M^2 g - \frac{1}{2}q_\phi, \quad q_\phi(X,X) = ||X\lrcorner\phi||_M^2.$$

Moreover, $(\widetilde{M}^{3,1}, \tilde{g})$ is Einstein with Einstein constant $-\|\phi\|^2/6$.

Example. Consider the product $(M^7 := Q^3 \times P^4, g = g_Q + g_P)$ of 3-dimensional Riemannian manifold (Q^3, g_Q) with a 4-dimensional Riemannian manifold (P^4, g_P).

• Assume that $\phi := \mathrm{vol}_Q$ is a special 3-form, where vol_Q is the is volume 3-form on the first factor, with $\|\phi\|^2 = \|\mathrm{vol}_Q\|^2 = 1$.

• Then $\langle X\lrcorner\mathrm{vol}_Q, Y\lrcorner\mathrm{vol}_Q\rangle = g_Q(X,Y)$ for any $X, Y \in \Gamma(TM^7)$ and

$$\mathrm{Ric}^g = \frac{1}{3}g - \frac{1}{2}g_Q.$$

$\implies \mathrm{Ric}^{g_Q} = -\frac{1}{6}g_Q$ and $\mathrm{Ric}^{g_P} = \frac{1}{3}g_P$.

• If the initial metric g is complete, then Q is a complete space of constant negative curvature (i.e. a quotient $\mathbb{R}H^3/\Gamma$ of the Lobachevski space $\mathbb{R}H^3$ by a lattice) and P is a compact Einstein 4-manifold. Note that the manifold M^7 is compact if Γ is a co-compact lattice.

• We get decomposable supergravity background of Type II, with internal space $M^7 = Q^3 \times P^4$ and space-time any Lorentzian Einstein 4-manifold $\widetilde{M}^{3,1}$ with Einstein constant $-1/6$.

• *Case C :* The SUGRA Einstein equation remains unchanged

Some material of G_2-structures

- A G_2-structure on M^7 is a reduction of the structure group SO_7 of the principal bundle of orthonormal frames $\mathrm{SO}(M, g)$ to G_2.

- A manifold M^7 admits a G_2 structure if and only if it is **orientable** and **spin**.

- G_2-structures are in bijective correspondence with *generic* or *stable* 3-forms $\omega \in \Lambda^3_+(M) \subset \Lambda^3(M)$

$$\omega := e^{127} + e^{347} + e^{567} + e^{135} - e^{245} - e^{146} - e^{236}.$$

- Since $\mathrm{G}_2 \subset \mathrm{SO}_7$, any G_2- structures defines an orientation and a Riemannian metric $g = g_\omega$:

$$g_\omega(X, Y)\,\mathrm{vol}_M := -\frac{1}{6}(X \lrcorner \omega) \wedge (Y \lrcorner \omega) \wedge \omega,$$

- A G_2 -structure is called *parallel* or *integrable*, if

$$\nabla^g \omega = 0 \quad \Leftrightarrow \quad \mathrm{d}\,\omega = \mathrm{d} *_7 \omega = 0.$$

- A G_2 -structure is called *weak* $-\,\mathrm{G}_2$ or *nearly parallel*, if

$$\mathrm{d}\,\omega = f *_7 \omega\,, \quad \text{for some} \quad \mathbb{R}^* \ni f = \text{cons.}$$

$(4,7)$-decomposable SUGRA solutions associated to weak G_2-structures

- Any weak G_2-structure admits real Killing spinors, hence is an Einstein manifold. This implies

Theorem. Let $\mathscr{M}^{10,1}$ be the oriented Lorentzian manifold given by the product of a four-dimensional oriented Lorentzian manifold $(\widetilde{M}^{3,1}, \tilde{g})$ with volume form $\mathrm{vol}_{\widetilde{M}}$ and a seven-dimensional oriented manifold M^7 admitting a G_2-structure $\phi \in \Omega^3_+(M)$, such that $\|\phi\|^2 = 7$. Define

$$\mathscr{F}^4_\pm := \pm 2\,\mathrm{vol}_{\widetilde{M}} + \star_7 \phi.$$

Then $(\mathscr{M}, g_{\mathscr{M}} = \tilde{g} + g, \mathscr{F}^4_\pm)$, where g is the Riemannian metric on M corresponding to ϕ, gives rise to a pair of $(4,7)$-decomposable supergravity backgrounds if and only if (M^7, ϕ) is a weak G_2-manifold and $(\widetilde{M}^{3,1}, \tilde{g})$ is Lorentz Einstein with negative Einstein constant $\Lambda := -15/6$.

A non-existence result

Proposition. If $\lambda = 0$ and $\phi := \star_7 F$ is a stable 3-form on M^7, where $F^4 \in \Omega^4_{\mathrm{cl}}(M^7)$, then the Maxwell equation for the flux form

$$\mathscr{F}^4 := F^4,$$

implies that **ϕ is ∇^g-parallel**, i.e. ϕ induces a parallel G_2-structure on M^7. In this case, the product

$$(\mathscr{M}^{11} = \widetilde{M}^{3,1} \times M^7, g_{\mathscr{M}} = \tilde{g} + g, F^4)$$

does not provides us with a $(4,7)$-decomposable supergravity background.

Invariant special 3-forms

$\Rightarrow$ One can separate the examination of Type C invariant special 3-forms into the following two subclasses:

- Type Cα, i.e. $\phi := \star_7 F$ is an invariant **generic 3-form** and thus it induces a homogeneous co-callibrated *weak* G_2*-structure* on $M^7 = G/H$.
- Type Cβ, i.e. $\phi := \star_7 F$ is an invariant **non-generic 3-form** on $M^7 = G/H$.

Remarks:

$\Rightarrow$ classification of homogeneous weak G_2-mnfds: [Friedrich et al-1998]

$\Rightarrow$ classification of homogeneous Lorentzian Einstein mnfds: [Komrakov Jnr-2001]

Invariant special 3-forms of Type Cα

Compact homogeneous G_2-manifolds & weak G_2-manifolds

Invariant (non-weak) G_2-structures	Invariant weak G_2-structures
T^7	$W_{k,l} = \dfrac{\mathrm{SU}_3}{\mathrm{S}^1_{k,l}}$
$\mathrm{S}^3 \times \mathrm{T}^4$	$\mathbb{V}_{5,2} = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_3^{\mathrm{st}}}$
$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 = \dfrac{\mathrm{SU}_2 \times \mathrm{SU}_2 \times \mathrm{S}^1}{\{e\}}$	$\mathrm{S}^7 = \dfrac{\mathrm{Sp}_2}{\mathrm{Sp}_1} = \dfrac{\mathrm{Sp}_2 \times \mathrm{U}_1}{\mathrm{Sp}_1 \times \mathrm{U}_1} = \dfrac{\mathrm{Sp}_2 \times \mathrm{Sp}_1}{\mathrm{Sp}_1 \times \mathrm{Sp}_1} = \dfrac{\mathrm{Spin}_7}{\mathrm{G}_2}$
$\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^1 = \dfrac{(\mathrm{SU}_2 \times \mathrm{SU}_2)}{\Delta\,\mathrm{SU}_2} \times \dfrac{(\mathrm{SU}_2 \times \mathrm{SU}_2)}{\Delta\,\mathrm{SU}_2} \times \mathrm{S}^1$	$B^7 = \dfrac{\mathrm{SO}_5}{\mathrm{SO}_3^{\mathrm{irr}}}$
$\mathrm{S}^6 \times \mathrm{S}^1$	$M_{k,l,m} := \dfrac{(\mathrm{S}^3 \times \mathrm{S}^3 \times \mathrm{S}^3)}{(\mathrm{S}^1 \times \mathrm{S}^1)}$
$\mathbb{V}^{4,2} \times \mathrm{T}^2$	$N_{k,l} = \dfrac{(\mathrm{SU}_3 \times \mathrm{SU}_2)}{(\mathrm{SU}_2 \times \mathrm{S}^1)}$
$\mathrm{S}^5 \times \mathrm{T}^2$	$W_{1,1} = \dfrac{\mathrm{SU}_3 \times \mathrm{SU}_2}{\mathrm{SU}_2^c \times \mathrm{U}_1}$
$\mathbb{F}_{1,2} \times \mathrm{S}^1$	
$\mathbb{C}P^3 \times \mathrm{S}^1$	

Invariant special 3-forms of Type Cβ

- We classify all compact almost effective homogeneous 7-manifolds $M^7 = G/H$ of a compact Lie group G $\Rightarrow$ For this step we proceed in terms of representation theory.

- We then focus on spaces which admits invariant 3-forms which are not stable.

- We find examples of spaces which admit **non-stable special 3-forms**:

$\rightsquigarrow$ $M^7 = \mathbb{CP}^2 \times \mathrm{S}^3 = (\mathrm{SU}_3 \,/\, \mathrm{U}_2) \times \mathrm{SU}_2$ (non-spin)

$\rightsquigarrow$ $M^7 = \mathrm{S}^3 \times \mathrm{T}^4 = \mathrm{SU}_2 \times \mathrm{T}^4$ (spin)

However: These examples do not provides us with special gravitational Einstein 7-manifolds. So, these special 3-forms do not induce supergravity backgrounds

Open problems for further research

In the presence of **internal fluxes**, i.e. $\mathscr{F} = \widetilde{F} + F$, the existence of Killing spinors requires some *special* G-structure (or *non integrable*):

- $\mathscr{M}^{10,1} = \widetilde{M}^{2,1} \times M^8$, where M^8 Spin_7 −structure
- $\mathscr{M}^{10,1} = \widetilde{M}^{4,1} \times M^6$, where M^6 nearly Kähler mnfd
- $\mathscr{M}^{10,1} = \widetilde{M}^{5,1} \times M^5$, where M^5 has a good contact metric structure, e.g. Sasakian mnfd

Thank You!