# Lecture III: Geometric Constructions Relating Different Special Geometries II

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# Plan of the third lecture

- One-loop quantum correction
- HK/QK-correspondence
- Special geometry of Euclidean $N = 2$ theories

# Some references for Lecture III

## Collaborations concerning HK/QK, 1-loop etc.


[CD] C.–, David, arXiv:1706.05516

[CS] C.–, Saha (MZ ‘17)

[CDiM] C.–, Dieterich, Mohaupt (LMP ‘17)

[ACDM] Alekseevsky, C.– , Dyckmanns, Mohaupt (JGP ‘15), arXiv:1305...

[ACM] Alekseevsky, C.– , Mohaupt (CMP ‘13), arXiv:1205...


## Related work


[MS] Macía, Swann (CMP ‘15), arXiv:1404...

[H] Hitchin (CMP ‘13), arXiv:1210...

[APP] Alexandrov, Persson, Pioline (JHEP ‘11).

[Ha] Haydys (JGP ‘08).

[RSV] Robles-Llana, Saueressig, Vandoren (JHEP ‘06).

# Some references for Lecture III

## Collaborations related to special geometry of Euclid. theories

[CDMV] C.–, Dempster, Mohaupt, Vaughan (JHEP ‘15)
[CDM] C.–, Dempster, Mohaupt (JHEP ‘14)
[CM] C.–, Mohaupt (JHEP ‘09).
[C] C.– (MS ‘06)
[CMMS2] C.–, Mayer, Mohaupt, Saueressig (JHEP ‘05).
[AC] Alekseevsky, C.– (AMST ‘05)
[ABCV] Alekseevsky, Blazic, C.–, Vukmirovic (JGP ‘05)
[CMMS1] C.–, Mayer, Mohaupt, Saueressig (JHEP ‘04).

## Related work

[DV] Dyckmanns, Vaughan (JGP ‘17)

# One-loop correction of the FS-metric I

Consider the FS-metric associated with a PSK domain $\bar{M}$. The following symmetric tensor field is called one loop correction of the FS-metric [RSV]:

$$
\begin{aligned}
g_{FS}^{c} &= \frac{\phi + c}{\phi} g_{\bar{M}} + \frac{1}{4\phi^2}\frac{\phi + 2c}{\phi + c} d\phi^2 \\
&\quad + \frac{1}{4\phi^2}\frac{\phi + c}{\phi + 2c}\left(d\tilde{\phi} + \sum(\zeta^j d\tilde{\zeta}_j - \tilde{\zeta}_j d\zeta^j) + ic(\bar{\partial} - \partial)\mathcal{K}\right)^2 \\
&\quad + \frac{1}{2\phi}\sum dq_a \hat{g}^{ab} dq_b + \frac{2c}{\phi^2} e^{\mathcal{K}} \left|\sum (X^j d\tilde{\zeta}_j + F_j(X) d\zeta^j)\right|^2,
\end{aligned}
$$

where $c \in \mathbb{R}$, $X^j = z^j/z^0$ and

$$
\mathcal{K} = -\log\left(\sum X^i N_{ij} \bar{X}^j\right)
$$

is the Kähler potential for the projective special Kähler metric $g_{\bar{M}}$.

# One-loop correction of the FS-metric II

## Theorem [ACDM]

For $c \geq 0$, the one loop correction $g^c_{FS}$ defines a 1-parameter family of quaternionic Kähler metrics on $\bar{N} = \bar{M} \times G$ deforming the FS-metric $g_{FS} = g^0_{FS}$.

## Sketch of proof

- Applying the rigid c-map to the underlying CASK mf. $M$ we obtain a pseudo-HK mf. $N$.
- The $\nabla$-horizontal lift of $2J\xi$ defines a Killing v.f. $Z$ on $N$ satisfying the assumptions of the HK/QK-correspondence explained on the next slides.
- Applying the HK/QK-correspondence yields a 1-parameter family of pseudo-QK metrics, of which we determine the domain of positivity.
- Finally we check that this family coincides with the one loop correction of the FS-metric. $\square$

# The HK/QK-correspondence I

The following result generalizes work of Haydys [Ha]:

## Theorem [ACM]

- Let $(M, g, J_1, J_2, J_3)$ be a pseudo-HK mf. with a timelike or spacelike Killing v.f. $Z$ s.t.
  - $\mathcal{L}_Z J_1 = 0$, $\mathcal{L}_Z J_2 = -2J_3$,
  - $\exists f : df = -\omega_1(Z, \cdot)$, $\omega_1 = g(J_1 \cdot, \cdot)$,
  - $f$ and $f_1 := f - g(Z, Z)/2$ are nowhere zero.

  Then from the data $(M, g, J_1, J_2, J_3, f)$ one can construct a pseudo-QK mf. $(M', g')$ with $\dim M' = \dim M$. The signature of $g'$ depends only on that of $g$ and the signs of $f$ and $f_1$.
- Cases when $g' > 0$:
- $g' > 0$ of $Ric > 0$ if $g > 0$ and $f_1 > 0$ and
- $g' > 0$ of $Ric < 0$ if either:
  $g > 0$ and $f < 0$ or
  $g$ has signature $(4k, 4)$, $f < 0$ and $f_1 > 0$.

# The HK/QK-correspondence II

## Remarks

- In [ACDM] we give a simple explicit formula for the QK-metric $g'$ obtained from the HK/QK-correspondence:

$$g' = \frac{1}{2|f|}\tilde{g}_P|_{M'}, \quad \tilde{g}_P := g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2,$$

- where $P \to M$ is an $S^1$-principal bundle with connection $\eta$ and curvature $\omega_1 - \frac{1}{2}d\beta$, $\beta = gZ$, endowed with

$$\begin{aligned} g_P &= \frac{2}{f_1}\eta^2 + g, \\ \theta_0^P &= \frac{1}{2}df,\ \theta_1^P = \eta + \frac{1}{2}\beta,\ \theta_2^P = \frac{1}{2}\omega_3 Z,\ \theta_3^P = -\frac{1}{2}\omega_2 Z, \end{aligned}$$

- and $M' \subset P$ is transversal to $Z_1^P = \tilde{Z} + f_1 X_P$.

# The HK/QK-correspondence III

## Remarks (continued)

- Using this formula, we check that rigid c-map metric is mapped to 1-loop corrected sugra c-map metric by this correspondence.
- Similar result obtained in [APP] by applying twistor methods and the inverse construction, the QK/HK-correspondence.
- Simplest case is $\bar{M} = \{pt\} \to$ 1-param. defo of $\mathbb{C}H^2$ by explicit complete QK metrics, see next slides. (Full domain of positivity of 1-loop correction has also components with incomplete metric, including one found by Haydys [Ha].)
- This example of the HK/QK-correspondence is also discussed in [H], but without the QK metric.
- $\exists$ similar K/K-correspondence [ACM,ACDM] and a version in generalized geometry [CD]. $\to$ related to Swann's twist [MS]
- $\exists$ ASK/PSK-corresp. relating rigid and sugra r-map [CDiM].

# Simplest example of a one-loop deformed QK metric: deformation of the universal hypermultiplet

### Example

For $\bar{M} = pt$, i.e. $F = \frac{i}{2}(z^0)^2$, we have:

$$
\begin{aligned}
g^c &= \frac{1}{4\phi^2}\left(\frac{\phi+2c}{\phi+c}d\phi^2 + \frac{\phi+c}{\phi+2c}(d\tilde{\phi}+\zeta^0 d\tilde{\zeta}_0 - \tilde{\zeta}_0 d\zeta^0)^2 \right. \\
&\quad \left. +2(\phi+2c)((d\tilde{\zeta}_0)^2+(d\zeta^0)^2)\right),
\end{aligned}
$$

with $g^0$ the complex hyperbolic plane metric and $g^c$ complete for $c \geq 0$.

# Some properties of the one-loop deformed UHM, see [CS]

- Family $g^c$ interpolates between the complex hyperbolic metric $g^0$ and real hyperbolic metric.
- To see this we re-parametrize $c = 1/b$ and $(\phi, \tilde{\phi}, \zeta^0, \tilde{\zeta}_0) = (\phi', \tilde{\phi}', \sqrt{b}\,\zeta'^0, \sqrt{b}\,\tilde{\zeta}'_0)$, obtaining

  $$h^b = \frac{1}{4\phi'^2}\left[\frac{b\phi'+2}{b\phi'+1}\,\mathrm{d}\phi'^2 + \frac{b\phi'+1}{b\phi'+2}(\mathrm{d}\tilde{\phi}' + b\zeta'^0\mathrm{d}\tilde{\zeta}'_0 - b\tilde{\zeta}'_0\mathrm{d}\zeta'^0)^2 \right.$$
  $$\left. +2(b\phi'+2)\left((\mathrm{d}\tilde{\zeta}'_0)^2 + (\mathrm{d}\zeta'^0)^2\right)\right],$$

  where $b > 0$. Now the family can be extended to $b = 0$.
- The metric $h^0$ has constant negative curvature.
- Conformal structure at infinity acquires pole for $b > 0$.
- The metric $g^c$ ($c > 0$) is not only Einstein and half-conformally flat but of negative curvature and
- quarter-pinched: $\frac{1}{4} < \delta_p < 1$ (limits attained as $\phi \to \infty$, 0).

# Special geometry of Euclidean supersymmetry

## Special geometries of $N = 2$ Euclidean vector multiplets [CMSS1,CMMSS2,CM,CDMV]

| d | susy | sugra |
|---|---|---|
| 4 | affine special para-Kähler | projective special para-Kähler |
| 3 | para-hyper-Kähler | para-quaternionic Kähler |

## Definition

- A para-Kähler manifold $(M, g, J)$ is a pseudo-Riem. mf. $(M, g)$ endowed with a parallel skew-symmetric endomorphism field $J$ s.t. $J^2 = \mathbb{1}$.
- A para-hyper-Kähler manifold $(M, g, J_1, J_2, J_3)$ is a pseudo-Riem. mf. $(M, g)$ endowed 3 parallel skew-symm. endom. fields $J_1$, $J_2$, $J_3 = J_1 J_2 = -J_2 J_1$ s.t. $J_1^2 = J_2^2 = \mathbb{1}$.

# Para-quaternionic Kähler manifolds

## Definition

(i) An almost para-quaternionic structure on a manifold $M$ is a subbundle $Q \subset \mathrm{End}\, TM$ s.t. $\forall p \in M \; \exists$ basis $(I, J, K = IJ = -JI)$ of $Q_p$ such that $I^2 = J^2 = \mathbb{1}$.

(ii) Let $\dim M > 4$. A para-quaternionic Kähler structure on $M$ is a pair $(g, Q)$ consisting of a pseudo-Riem. metric and a parallel para-quat. structure $Q \subset \mathfrak{so}(TM)$. The triple $(M, g, Q)$ is called a para-quaternionic Kähler (para-QK) manifold.

## Remarks

- If $\dim M = 4$, in (ii) one has to require in addition $Q \cdot R = 0$.
- para-QK $\Longrightarrow$ Einstein.
- para-HK $\Longrightarrow$ para-QK and $Ric = 0$.
- $\exists$ classification of symm. para-QK mfs. with $Ric \neq 0$ [AC] and cont. families of symm. para-HK mfs. of np. gps. [ABCV,C].

# Symmetric para-quaternionic Kähler manifolds I

## Theorem [AC]

The following exhausts all s.c. symm. para-QK mfs. with $Ric \neq 0$ of classical groups:

A)

$$\frac{SL(n+2,\mathbb{R})}{S(GL^+(2,\mathbb{R}) \times GL^+(n,\mathbb{R}))}, \quad \frac{SU(p+1,q+1)}{S(U(1,1) \times U(p,q))},$$

BD)

$$\frac{SO_0(p+2,q+2)}{SO_0(2,2) \times SO_0(p,q)}, \quad \frac{SO^*(2n+4)}{SO^*(4) \times SO^*(2n)},$$

C)

$$\frac{Sp(\mathbb{R}^{2n+2})}{Sp(\mathbb{R}^2) \times Sp(\mathbb{R}^{2n})},$$

# Symmetric para-quaternionic Kähler manifolds II

## Theorem [AC]

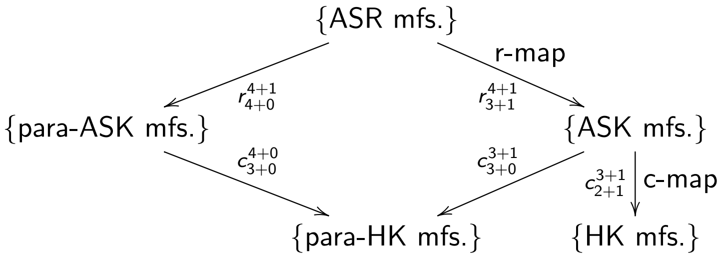
The following exhausts all s.c. symm. para-QK mfs. with $Ric \neq 0$ of exceptional groups:

$$\frac{E_{6(6)}}{SL(2,\mathbb{R})\times SL(6,\mathbb{R})}, \quad \frac{E_{6(2)}}{SU(3,3)\times SU(1,1)}, \quad \frac{E_{6(-14)}}{SU(5,1)\times SU(1,1)},$$
$$\frac{E_{7(7)}}{SL(2,\mathbb{R})\times Spin_0(6,6)}, \frac{E_{7(-5)}}{SL(2,\mathbb{R})\times SO^*(12)}, \frac{E_{7(-25)}}{SL(2,\mathbb{R})\times Spin_0(10,2)},$$
$$\frac{E_{8(8)}}{SL(2,\mathbb{R})\times E_{7(7)}}, \quad \frac{E_{8(-24)}}{SL(2,\mathbb{R})\times E_{7(-25)}},$$
$$\frac{F_{4(4)}}{SL(2,\mathbb{R})\times Sp(\mathbb{R}^6)},$$
$$\frac{G_{2(2)}}{SO_0(2,2)}.$$

# Euclidean versions of the rigid r- and c-map

## Theorem [CMSS1-2]

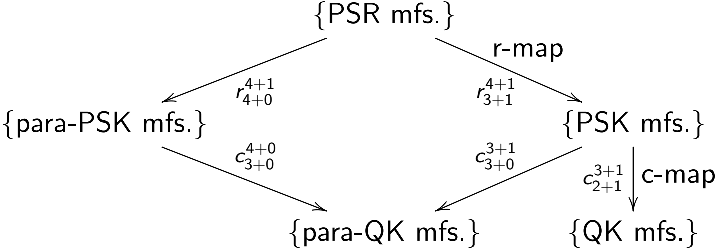
- ∃ construction $r^{4+1}_{4+0}$ (temporal $r$-map) which associates a para-ASK mf. with every ASR mf.
- ∃ construction $c^{3+1}_{3+0}$ (temporal $c$-map) which associates a para-HK mf. with every ASK mf.
- ∃ construction $c^{4+0}_{3+0}$ (Euclidean $c$-map) which associates a para-HK mf. with every para-ASK mf.
- The resulting diagram commutes up to isometry:



# Euclidean versions of the supergravity r- and c-map

## Theorem [CM,CDMV]

- Dimensional reduction of supergravity coupled to $N = 2$ vector multiplets induces constructions summarized in the following diagram:




## Open problem

- Does the diagram commute, up to isometry?

# Example: reduction of pure 5-dim. supergravity

## Theorem [CDM]

- Applying the the supergravity constructions $c^{3+1}_{3+0} \circ r^{4+1}_{3+1}$ and $c^{4+0}_{3+0} \circ r^{4+1}_{4+0}$ to the PSR mf. $\mathcal{H} = \{pt\}$ yields 2 different open orbits of the solvable Iwasawa subgroup of $G_{2(2)}$ on the para-QK symmetric space $G_{2(2)}/SO_0(2,2)$.
- In particular, the resulting manifolds are locally isometric to each other.

# Temporal and Euclidean supergravity c-maps via HK/QK

### Theorem [DV]

- The temporal and Euclidean supergravity c-maps and a one-parameter deformation thereof can be obtained from suitable generalizations of the HK/QK-correspondence.