Lecture IV: Completeness Results

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Some references for Lecture IV

[CDJL] C.–, Dyckmanns, Juengling, Lindemann, math.DG:1701.7882
[CDS] C.–, Dyckmanns, Suhr (Springer INdAM '17)
[CNS] C.–, Nardmann, Suhr (CAG '16) (PLMS '14)
[CHM] C.–, Han, Mohaupt (CMP '12).

Plan of the fourth lecture:

- Motivation
- Completeness of PSR mfs.
- Completeness of PSK mfs.

Main idea

Use supergravity constructions (and one-loop deformation) to obtain new complete quaternionic Kähler manifolds

- Recall:

Theorem [CHM]

(i) The supergravity r-map maps complete PSR mfs. $\mathcal{H}$ to complete PSK mfs. $\bar{M}$.

(ii) The supergravity c-map maps complete PSK mfs. $\bar{M}$ to complete QK mfs. $\bar{N}$.

Problems

- Control completeness of the initial PSR or PSK manifold.
- Control completeness under the one-loop deformation.

Completeness of centroaffine hypersurfaces

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a centroaffine hypersurface with positive definite centroaffine metric g.

We are interested in the relation between

1) closedness,
2) Euclidian completeness and
3) completeness (with respect to g).

Under natural assumptions:
3) $\implies$ 1) $\iff$ 2).

Main problem:

Prove that 1) $\implies$ 3) in some interesting cases.

Example: Theorem (Cheng and Yau, CPAM '89)

1) $\implies$ 3) if $\mathcal{H}$ is an affine sphere, i.e. if $\nabla^g \nu = 0$.

Completeness of higher dimensional PSR manifolds

Theorem [CNS]

A PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$ is complete if and only if $\mathcal{H} \subset \mathbb{R}^{n+1}$ is closed.

Corollary

Let $\mathcal{H}$ be a locally strictly convex component of the level set $\{h = 1\}$ of a homogeneous cubic polynomial h on $\mathbb{R}^{n+1}$. Then $\mathcal{H}$ defines a complete quaternionic Kähler metric of negative scalar curvature on $\mathbb{R}^{4n+8}$.

Applications

Using the Corollary we can construct many new explicit complete QK manifolds and even families depending on an arbitrary number of parameters, including multi-parameter defos of symm. spaces [CDJL], as will be shown in the next lecture.

Completeness of centroaffine hypersurfaces

Open problem

Does the theorem extend to (definite) centroaffine hypersurfaces defined by homogeneous polynomials h of higher degree?

State of the art:

1. It holds for generic polynomials.
2. It does not hold for general (real analytic) functions.

Sketch of proof of the main theorem I

- Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidian complete centroaffine hypersurface with positive definite centroaffine metric g.
- We have to show that $\mathcal{H}$ is complete if $\mathcal{H} \subset \{h = 1\}$ for a homogeneous cubic polynomial h. Let us not assume this yet.
- Consider the open cone $U = \mathbb{R}^{>0} \cdot \mathcal{H} \subset \mathbb{R}^{n+1}$ and let $k \in \mathbb{R}^*$.

Lemma 1

- There exists a unique smooth homogeneous function $h : U \to \mathbb{R}$ of degree k such that $h|_{\mathcal{H}} = 1$.
- For every hyperplane E tangent to $\mathcal{H}$ the intersection $B := U \cap E \subset E$ is a bounded convex domain.
- $$\varphi : B \to \mathcal{H}, \quad x \mapsto h(x)^{-1/k} x,$$

  is a parametrization of $\mathcal{H}$.

Sketch of proof of the main theorem II

Lemma 2

In the above parametrization the centroaffine metric is given

$$g = -\frac{1}{k\bar{h}}\partial^2\bar{h} + \frac{k-1}{(k\bar{h})^2}d\bar{h}^2,$$

where $\bar{h}$ denotes the restriction of h to B and ∂ denotes the flat connection of the affine space $E \supset B$.

Lemma 3

Let $k > 0$. Assume that there exists $\varepsilon \in (0, k)$ such that $f = \sqrt[k-\varepsilon]{\bar{h}}$ is concave. Then $\mathcal{H}$ is complete.

Sketch of pf. of Lemma 3

A calculation shows

$$g = \frac{k-\varepsilon}{f}\left(-\frac{1}{k}\partial^2 f\right) + \frac{\varepsilon}{(k-\varepsilon)(k\bar{h})^2}d\bar{h}^2 \geq \underbrace{\frac{\varepsilon}{k^2(k-\varepsilon)}}_{C:=}(d\ln\bar{h})^2.$$

Sketch of proof of the main theorem III

Let $\gamma : I = [0, T) \to B,\ T \in (0, \infty]$, be a curve which is not contained in any compact subset of B and $I \ni t_i \to T$.

- Then $h(\gamma(t_i)) \to 0$ and the previous estimate implies

$$\begin{aligned} L(\gamma) &\geq L(\gamma|_{[0,t_i]}) \geq \sqrt{C} \int_0^{t_i} \left| \frac{d}{dt} \ln h \circ \gamma \right| dt \\ &\geq \sqrt{C} \left| \int_0^{t_i} \frac{d}{dt} \ln h \circ \gamma dt \right| \\ &= \sqrt{C} |\ln h(\gamma(t_i)) - \ln h(\gamma(0))| \to \infty \end{aligned}$$

This finishes the proof of Lemma 3. □

Sketch of proof of the main theorem IV

Lemma 4

If h is a cubic polynomial then $\sqrt{h}$ is concave

Lemma 4 shows that the assumptions of Lemma 3 are satisfied with $(k, \epsilon) = (3, 1)$. This finishes the proof of the main theorem. □

Proof of Lemma 4

- Consider a line $x + tv$ in E with $x \in B$. Its intersection with B corresponds to the segment $t \in (a, b)$:
- We check that $h_0(t) = h(x + tv)$ satisfies $\sqrt{h_0}'' \leq 0$ on (a, b).
- $$4h_0^{3/2}\sqrt{h_0}'' = 2h_0 h_0'' - (h_0')^2 =: f, \quad f' = 2h_0 h_0'''$$
- Since $h_0 > 0$ on (a, b) and h_0''' is constant, this shows that f is monotone. $\implies$ its values lie between $f(a) \leq 0$ and $f(b) \leq 0$.

□

Further results (about general centroaffine hypersurfaces): The canonical Lorentzian metric on the open cone U

Proposition

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be any Euclidian complete centroaffine hypersurface with positive definite centroaffine metric , $k > 1$ and h the corresponding homogeneous function of degree k. Then

$$g_L := -\frac{1}{k}\partial^2 h$$

is a Lorentzian metric on U, which is globally hyperbolic iff $\mathcal{H}$ is complete.

Further results: Regular boundary behaviour implies completeness

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be any Euclidian complete centroaffine hypersurface with positive definite centroaffine metric. We assume that $k > 1$ and that h extends to a smooth homogeneous function $h : V \to \mathbb{R}$ defined on some open subset $V \subset \mathbb{R}^{n+1}$ such that $\overline{U} \setminus \{0\} \subset V$.

Definition

Under the above assumptions, we say that the hypersurface $\mathcal{H}$ has regular boundary behaviour if

(i) $dh_p \neq 0$ for all $p \in \partial U \setminus \{0\}$. In particular, $\partial U \setminus \{0\}$ is smooth.

(ii) $-\partial^2 h$ is positive semi-definite on $T(\partial U \setminus \{0\})$ with only one-dimensional kernel.

Regular boundary behaviour implies completeness

Theorem [CNS]

Let $\mathcal{H} \subset \mathbb{R}^{n+1}$ be a Euclidian complete centroaffine hypersurface with regular boundary behaviour. Then $\mathcal{H}$ is complete.

Regular boundary behaviour is generic

- Let $V \subset \mathbb{R}^{n+1}$ be an open subset and $k > 1$.
- Denote by $\mathcal{F}(V,k) \subset C^\infty(V)$ the set of homog. fcts. h of deg. k s.t. $\exists$ open cone $U \subset V$ s.t. $\overline{U} \setminus \{0\} \subset V$ and s.t.

$$\mathcal{H}(h,U) := \{p \in U | h(p) = 1\}$$

  is Euclidian complete with $g > 0$.
- Put
  $\mathcal{F}_{reg}(V,k) := \{h \in \mathcal{F} | \mathcal{H}(h,U)$ has reg. bdry. beh. for some $U\}$.

Theorem [CNS]

$\mathcal{F}_{reg}(V,k) \subset \mathcal{F}(V,k)$ is open and dense (in the Fréchet topology).

Regular boundary behaviour is generic: Case of polynomial functions

- Denote by

$$\mathcal{P}(k) \subset \mathcal{F}(\mathbb{R}^{n+1}, k), \quad \mathcal{P}_{reg}(k) \subset \mathcal{F}_{reg}(\mathbb{R}^{n+1}, k)$$

the subsets consisting of polynomial functions.

Theorem [CNS]

$\mathcal{P}_{reg}(k) \subset \mathcal{P}(k)$ is open and dense.

Completeness of projective special Kähler manifolds I

Definition

A CASK manifold (M, J, g, ∇, ξ) is said to have regular boundary behaviour if it admits an embedding $i\colon M \to \mathcal{M}$ into a mf. with boundary $\mathcal{M}$ s.t. $i(M) = \mathcal{M} \setminus \partial\mathcal{M}$ and the tensor fields (J, g, ξ) smoothly extend to $\mathcal{M}$ s.t. $\forall p \in \partial\mathcal{M}$: $f(p) = 0$, $df_p \neq 0$ and $g_p \leq 0$ on $\mathcal{H}_p := T_p\partial\mathcal{M} \cap J(T_p\partial\mathcal{M})$ with kernel $\mathrm{span}\{\xi_p, J\xi_p\}$, where $f = g(\xi, \xi)$.

Definition

We will assume that $\xi, J\xi$ generate a principal $\mathbb{C}^*$-action on $\mathcal{M}$ with compact quotient $\bar{\mathcal{M}} := \mathcal{M}/\mathbb{C}^*$. Then the interior $\bar{M} = \bar{\mathcal{M}} \setminus \partial\bar{\mathcal{M}}$ is called a PSK manifold with regular boundary behaviour.

Theorem [CDS]

Projective special Kähler manifolds with regular boundary behaviour are complete.

Completeness of projective special Kähler manifolds II

Sketch of proof

- Consider underlying CASK mf. (M, J, g, ∇, ξ), $M = \mathcal{M} \setminus \partial\mathcal{M}$.
- Step 1: $\forall p \in \partial\mathcal{M} : g_p$ is nondeg. of signature $(2, 2n)$.
- Consider $\gamma : [0, b) \to \bar{M}$ not contained in any cp. subset, $0 < b \leq \infty$. $\implies$ γ has an accumulation pt. $\bar{p}_0 \in \partial\bar{\mathcal{M}}$.
- Case 1: γ has no other accumulation pt.
- Case 2: γ has a 2nd accum. pt.
- Case 1 $\implies$ $\forall$ nbh. of $\bar{p}_0$ $\exists a \in (0, b) : \gamma([a, b)) \subset$ that nbh.
- By Step 1, $\exists$ cx. hypersurf. $\mathcal{N} \subset \mathcal{M}$ through $p_0 \in \pi^{-1}(\bar{p}_0)$ s.t. $-g > 0$ on $T\mathcal{N}$. On $N := \mathcal{N} \cap M$ the PSK metric $\bar{g}$ satisfies:
- $$(\pi^*\bar{g})|_{TN} = \left(-\frac{g}{f} + \frac{\alpha^2 + (J^*\alpha)^2}{f^2}\right)\Bigg|_{TN} \geq \frac{df^2}{4f^2}\Bigg|_{TN},$$

  where $\alpha = g(\xi, \cdot) = \frac{1}{2}df$. $\implies$ $L(\gamma) = \infty$.

Completeness of projective special Kähler manifolds III

Sketch of proof continued

- Case 2: Use the estimate

$$(\pi^*\bar{g})|_{TN} \geq -\frac{g}{f}\bigg|_{TN} \geq -\frac{1}{\epsilon}g\bigg|_{TN} =: g',$$

  which holds near $\bar{p}_0$, where $g' > 0$ (as a tensor on $\mathcal{N}$).
- Any curve with initial pt. in a small g'-ball $B_{\delta/2}(\bar{p}_0)$ and endpoint outside $B_\delta(\bar{p}_0)$ has length $\geq c > 0$.
- Since γ has accumulation points $\bar{p}_0 \neq \bar{p}_1$, there exists $\delta > 0$ such that $\bar{p}_1 \notin B_{3\delta/2}(\bar{p}_0)$.
- Then γ passes trough $B_{\delta/2}(\bar{p}_0)$ and leaves $B_\delta(\bar{p}_0)$ an arbitrarily large number k of times. $\implies L(\gamma) \geq kc \to \infty$. □