Lecture V: Construction of Complete Quaternionic Kähler Manifolds

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Some references for Lecture V

[CDJL] C.–, Dyckmanns, Juengling, Lindemann, math.DG:1701.7882

[CDS] C.–, Dyckmanns, Suhr (Springer INdAM '17)

[CDL] C.–, Dyckmanns, Lindemann (PLMS '14)

[CHM] C.–, Han, Mohaupt (CMP '12).

[L] LeBrun (Duke '91).

Plan of the fifth lecture:

- Completeness results for the one-loop deformation.
- Classification results for complete PSR mfs. and corresponding QK mfs.

Completeness of the one-loop corrected c-map metric

Theorem [CDS]

Let $(\bar{M}, \bar{g})$ be a PSK manifold with regular boundary behaviour. Then the corresponding one-loop deformation g^c_{FS} is a family of complete QK metrics for $c \geq 0$.

Consider the supergravity q-map, which is the composition of the supergravity r- and c-maps.

Theorem [CDS,CHM]

The one-loop deformed sugra q-map associates a family of complete QK manifolds of dim. $4n + 8$ (of $scal < 0$) depending on a parameter $c \geq 0$ with every complete projective special real manifold of dimension n.

$\Longrightarrow$ it is interesting to classify complete PSR manifolds

One-loop deformation of symmetric spaces

Corollary [CDS]

All known homog. QK mfs. of $scal < 0$ can be deformed in this way by complete QK manifolds, with exception of $\mathbb{H}H^n$.

Proof

- All of these homog. spaces with exception of $Gr_2(\mathbb{C}^{n+3})^*$ are in the image of the q-map.
- $Gr_2(\mathbb{C}^{n+3})^* = c(\mathbb{C}H^n)$ and the PSK manifold $\mathbb{C}H^n$ has regular boundary behaviour.

□

Remark

The quaternionic hyperbolic spaces $\mathbb{H}H^n$ were already known to admit deformations by complete QK metrics [L].

Classification of complete PSR curves and surfaces

Theorem [CHM]

There are only 2 complete PSR curves (up to equivalence):

i) $\{(x, y) \in \mathbb{R}^2 | x^2 y = 1,\ x > 0\}$,

ii) $\{(x, y) \in \mathbb{R}^2 | x(x^2 - y^2) = 1,\ x > 0\}$.

Theorem [CDL]

There are only 5 discrete examples and a 1-parameter family of complete PSR surfaces:

a) $\{(x, y, z) \in \mathbb{R}^3 | xyz = 1,\ x > 0, y > 0\}$,

b) $\{(x, y, z) \in \mathbb{R}^3 | x(xy - z^2) = 1,\ x > 0\}$,

c) $\{(x, y, z) \in \mathbb{R}^3 | x(yz + x^2) = 1,\ x < 0, y > 0\}$,

d) $\{(x, y, z) \in \mathbb{R}^3 | z(x^2 + y^2 - z^2),\ z < 0\}$,

e) $\{(x, y, z) \in \mathbb{R}^3 | x(y^2 - z^2) + y^3 = 1,\ y < 0, x > 0\}$,

f) $\{\cdots | y^2 z - 4x^3 + 3xz^2 + bz^3 = 1,\ z < 0, 2x > z\}$, $b \in (-1, 1)$.

- q-map $\rightarrow$ complete QK manifolds of co-homogeneity ≤ 2.

Classification of complete PSR manifolds with reducible cubic polynomial

Theorem[CDJL]

Every complete PSR manifold $\mathcal{H} \subset \{h = 1\} \subset \mathbb{R}^{n+1}$, $n \geq 2$, for which h is reducible is linearly equivalent to exactly one of the following:

a) $\{x_{n+1}(\sum_{i=1}^{n-1} x_i^2 - x_n^2) = 1,\ x_{n+1} < 0, x_n > 0\}$,

b) $\{(x_1 + x_{n+1})(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 + x_{n+1} < 0\}$,

c) $\{x_1(\sum_{i=1}^{n} x_i^2 - x_{n+1}^2) = 1, \quad x_1 < 0, x_{n+1} > 0\}$,

d) $\{x_1(x_1^2 - \sum_{i=2}^{n+1} x_i^2) = 1, \quad x_1 > 0\}$.

- Under the q-map, these are mapped to complete QK manifolds of co-homogeneity ≤ 1.
- The series d) is mapped to a series of complete QK manifolds of co-homogeneity 1.