The Teleparallel Trick

John Huerta
(Typeset by Henrik Winther)

March 16, 2018

This will be a theory of gravity, which Einstein was interested in when he tried (unsuccessfully) to unify the forces. If you want to see how to write general relativity in terms of a flat connection, this will show you how to do that.

0.1 Einstein gravity

Let (M^4, g) be of signature $(3, 1)$. Einstein gravity says that this manifold is ricci flat,

$$\mathrm{Ric}(g) = 0$$

A physicist would say that this is vacuum gravity with zero cosmological constant. The Ricci curvature is defined in terms of ∇^0, which is the unique torsion-free metric connection.

$$Xg(Y, Z) = g(\nabla^0_X Y, Z) + g(Y, \nabla^0_X Z)$$

Theorem 1. *Let T be any vector valued 2-form: $T : \Lambda^2 TM \to TM$. Then there exists a unique metric connection ∇ with torsion T.*

Proof. Idea of proof: Do the same calculation as to get the Levi-Civita connection, but include T. $\square$

0.2 Einstein's equation, v2

When you have an affine connection ∇, you can compute its curvature R, and take its trace to be $\mathrm{Ric}(\nabla)$. Then we can write

$$\begin{aligned} &\mathrm{Ric}(\nabla) = 0 \\ &T_\nabla = 0 \end{aligned}$$

Thus these equations are the same as Einstein's equations, but written without any metric.

0.3 The Teleparallel Trick

Suppose we have the space of connections, or metric connections, or connections on any bundle. This is an affine space. In the metric case, we have a favourite metric, ∇^0. The teleparallel trick is to "re-zero" the affine space. We are going to pick a flat connection ∇, and its torsion T_∇. Then

$$\nabla^{\text{flat}} = \nabla^0 + T$$
$$\nabla^0 = \nabla^{\text{flat}} - T$$

0.4 Einstein's equation v3

$$\mathrm{Ric}_{\nabla^{\text{flat}}-T} = 0 \tag{1}$$

Question 1. *What does it take to choose ∇^{flat}?*

Let $x \in U \subset M$. We need

$$TU \to^e U \to \mathbb{R}^3$$

The physicists call this a coframe field.

0.5 Teleparallel gravity

Let

$$TM \to^e \tau$$

be an isomorphism of vector bundles. τ is called the "fake tangent bundle". Let D be a metric compatible connection on τ.

$$g(X,Y) = \eta(e(X), e(Y))$$
$$e(\nabla_X Y) = D_X e(Y)$$

This is a standard structure in physics, instead of varying the metric, you fix the metric and vary the "coframe field". In fact this is necessary to do spin-gravity.

Question 2. *When is ∇ torsion-free?*

Proposition 1.

$$eT = d_D e$$

Proof.

$$eT(X,Y) = e\nabla_X Y - e\nabla_Y X - e[X,Y]$$
$$= D_X e(Y) - D_Y e(X) - e([X,Y]) = d_D e$$

□

0.6 Einstein's equation, v4

$$\mathrm{Ric}_\nabla = 0$$
$$d_D e = 0$$

Teleparallel gravity¿ Fix flat metric connection D_{flat} on τ. There exists a uniqur D^0 with properties

- $d_{D^0} e = 0$, i.e. torsion-freeness
- $D^0 = D^{\mathrm{flat}} - T$
- $\mathrm{Ric}_{D^{\mathrm{flat}} - T} = 0$

Proposition 2. *This is locally equivalent to Einstein's equation.*

Proof.

$$TU \to^e U \times \mathbb{R}^3$$

$D^{\mathrm{flat}} - T$ is Levi-Civita over this open set. □