Remez inequality and propagation of smallness for solutions of second order elliptic PDEs

Part I. Classical Remez inequality, analytic propagation of smallness

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Chebyshev polynomials

Definition

The n-th Chebyshev polynomial of the fist kind is the polynomial T_n of degree n which satisfies the identity

$$T_n(\cos t) = \cos nt.$$

Clearly $T_1(x) = x, \; T_2(x) = 2x^2 - 1$ and the trigonometric formula

$$\cos(n+1)t + \cos(n-1)t = 2\cos nt \cos t$$

implies the recursive formula for T_n

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Properties

- Leading coefficient: $T_n(x) = 2^{n-1}x^n + ... + c_n$
- Alternating min-max: We fix n and let $x_k = \cos(k\pi/n)$, $k = 0, ..., n$. Then $T_n(x_k) = (-1)^k$, $-1 = x_n < x_{n-1} < ... < x_k < ... < x_1 < x_0 = 1$.
- Extremal property:

$$\max_{-1 \le x \le 1} |T_n(x)| = 1 \le 2^{n-1} \max_{-1 \le x \le 1} |P_n(x)|$$

for any $P_n(x) = x^n + a_{n-1}x^{n-1} + ... + a_0$ (monic polynomial of degree n.)
- Formula $2T_n(x) = (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n$

Remez inequality

Theorem (Remez, 1936)

Let E be a measurable subset of an interval I and $|E| = m$. Then for any polynomial P_n of degree n

$$\max_{x\in I} |P_n(x)| \le T_n\left(1 + \frac{2(|I| - m)}{m}\right) \max_{x\in E} |P_n(x)|$$

The equality is attained when $P_n(x) = CT_n(2x/m)$, $I = (-m/2, m/2 + a)$ and $E = (-m/2, m/2)$.

Corollary

$$\max_{x\in I} |P_n(x)| \le \left(\frac{4|I|}{|E|}\right)^n \max_{x\in E} |P_n(x)|$$

Tool: Lagrange interpolation formula

If P is a polynomial of degree $\leq n$ and $y_0, ..., y_n$ are distinct points then

$$P(y) = \sum_{j=1}^{n} P(y_j) \prod_{k \neq j} \frac{y - y_k}{y_j - y_k}$$

Proof of Remez inequality

Renormalize to have $|E| = 2$, $I = [-1, 1 + a]$. Then find $y_j \in E$ such that $|y_j - y_k| \geq |x_j - x_k|$ (extremal points for Chebyshev polynomial) and $|1 + a - y_j| \geq |1 + a - x_j|$ and compare interpolation formulas for $P_n(1 + a)$ and $T_n(1 + a)$.

Turan-Nazarov inequality for exponential sums

Let $F_n(x) = \sum_{k=0}^n a_k e^{i\lambda_k x}$.

Theorem (Nazarov, 1993)

Let E be a measurable subset of an interval I and $|E| = m$.
(i) If all $\lambda_k \in \mathbf{R}$ then

$$\max_{x\in I} |F_n(x)| \leq \left(\frac{C|I|}{|E|}\right)^n \max_{x\in E} |F_n(x)|$$

(ii) If $\lambda_k \in \mathbf{C}$ we define $s = \max |Im\lambda_k|$, then

$$\max_{x\in I} |F_n(x)| \leq e^{s|I|} \left(\frac{C|I|}{|E|}\right)^n \max_{x\in E} |F_n(x)|$$

Reformulation of Remez inequality

The Remez inequality is equivalent to

$$|E| \leq 4|I| \left(\frac{\max_{x\in E} |P_n(x)|}{\max_{x\in I} |P_n(x)|} \right)^{1/n}.$$

We rewrite it as

$$|E_\delta| \leq 4|I|\delta^{1/n},$$

where

$$E_\delta = \{x \in I : |P_n(x)| < \delta \max_{x\in I} |P_n(x)|\}.$$

Classical results of Cartan and Polya

Let $P_n(z) = z^n + ...$ be a monic polynomial of degree n.

Lemma (Cartan, 1928)

Let $F_s = \{z \in \mathbf{C} : |P_n(z)| \le s^n\}$ *and let* $\alpha > 0$ *then there are disks* $B_j(z_j, r_j)$ *such that*

$$F_s \subset \cup_j B_j, \quad \sum_j r_j^\alpha \le e(2s)^\alpha$$

For $\alpha = 2$ one obtains an estimate for the measure of the set $|F_s|$. The sharp result here is due to Polya (1928) and it says that

$$|F_s| \le \pi s^2.$$

Hadamard three circle theorem

Theorem

Suppose that f is an analytic function in the domain $\{r_0 < |z| < R\}$. Let $M(r) = \max_{|z|=r} |f(z)|$ and $r_0 < r_1 < r_2 < r_3 < R$. Then

$$M(r_2) \le M(r_1)^{\alpha} M(r_3)^{1-\alpha}, \quad \textit{where } r_2 = r_1^{\alpha} r_3^{1-\alpha}.$$

It follows from the maximum principle for (sub)harmonic function $h(z) = \log |z^a f(z)|$. We have

$$r_2^a M(r_2) \le \max\{r_1^a M(r_1), r_3^a M(r_3)\}$$

and choose a such that $r_1^a M(r_1) = r_3^a M(r_3)$, then

$$r_2^a M(r_2) \le (r_1^a M(r_2))^{\alpha} (r_3^a M(r_3))^{1-\alpha}$$

Two-constant theorem

Theorem

Suppose that f is a bounded analytic function in a Jordan domain Ω such that $|f(z)| \leq M$ in Ω and $|f(\zeta)| \leq m$ when $\zeta \in E \subset \partial\Omega$. Then for any $z \in \Omega$

$$|f(z)| \leq m^{\omega_E(z)} M^{1-\omega_E(z)},$$

where $\omega_E(z)$ is the harmonic measure of E at point z.

In other words, ω_E is the harmonic function with boundary values 1 on E and 0 on $\partial\Omega \setminus E$. We once again use the maximum principle and compare $\log|f(z)|$ to $\omega_E(z)\log m + (1-\omega_E(z))\log M$.

Propagation of smallness for real analytic functions

Suppose that u is a real-analytic function in the unite ball $B \subset \mathbf{R}^d$, u extends to a holomorphic function U in $O \subset \mathbf{C}^d$ such that $O \cap \mathbf{R}^d \supset B$ and $|U| \le M$ in O. Suppose that $E \subset 1/2B$, $|E| > 0$ and $\max_E |u| \le m$. Then

$$\max_{1/2B} |u| \le C m^\beta M^{1-\beta},$$

where β depends on O and on $|E|$.

Theorem (Hayman, 1970)

Suppose that u is a harmonic function in B that satisfies $\max_B |u| \le M$. Then there exists a holomorphic function U in $B_{\mathbf{C}}(1/\sqrt{2})$ such that $U(x) = u(x)$ when $x \in B_{\mathbf{R}}(1/\sqrt{2})$ and $|U(z)| \le C(|z|)M$ when $|z| < 1/\sqrt{2}$.

Łojasiewicz inequality

Suppose that f is a non-zero real analytic function in $B \subset \mathbf{R}^n$, $Z_f = f^{-1}(0)$,. Then Z_f has dimension $n-1$, $Z_f = \cup_{j=0}^{n-1} A_j$, where A_j is a countable union of j-dimensional manifolds.

Let $Z_f \cap B_1 \neq \emptyset$. Then for any compact subset $K \subset B$ there exists $c > 0$ and β such that

$$|f(x)| \geq c \operatorname{dist}(x, Z_f)^\beta, \quad x \in K,$$

β is called the Łojasiewicz exponent of f (in K).

In particular

$$E_\delta = \{x \in K : |f(x)| < \delta \max_B |f|\} \subset K \cap (Z_f) + B(0, c_1 \delta^{1/\beta}),$$

where $D + B(0, \epsilon)$ is the ϵ-neighborhood of a set D.

Second order elliptic equations

We study operators of the form

$$Lf = div(A\nabla f),$$

where $A(x) = [a_{ij}(x)]_{1\le i,j\le d}$ is a symmetric matrix with Lipschitz entries and

$$\Lambda^{-1}\|v\|^2 \le (A(x)v, v) \le \Lambda\|v\|^2$$

uniformly in x.

We will study local properties of solutions to the equation $Lf = 0$ and changing the coordinates assume that L is a small perturbation of the Laplacian.

Harnack inequality and comparison of norms

Suppose that $Lf = 0$ in $B_1 \subset \mathbf{R}^d$ and $f \geq 0$ in B_1 then

$$\max_{B_{1/2}} f \leq C_H \min_{B_{1/2}} f.$$

In particular $E_\delta(f) = \{x \in B_{1/2} : |f(x)| < \delta \max_{B_{1/2}} |f|\}$ is empty when δ is sufficiently small.

We will also use the following inequality (equivalence of norms) for any solution f of $Lf = 0$ in B_1 we have

$$\frac{1}{|S_{1/2}|} \int_{S_{1/2}} |f|^2 \leq \max_{B_{1/2}} |f|^2 \leq C \frac{1}{|S_1|} \int_{S_1} |f|^2.$$