Remez inequality and propagation of smallness for solutions of second order elliptic PDEs

Part II. Logarithmic convexity for harmonic functions and solutions of elliptic PDEs

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Second order elliptic equations

We study operators of the form

$$Lf = div(A\nabla f),$$

where $A(x) = [a_{ij}(x)]_{1\le i,j\le d}$ is a symmetric matrix with Lipschitz entries and

$$\Lambda^{-1}\|v\|^2 \le (A(x)v, v) \le \Lambda\|v\|^2$$

uniformly in x.

We will study local properties of solutions to the equation $Lf = 0$ and changing the coordinates assume that L is a small perturbation of the Laplacian.

Harnack inequality and comparison of norms

Suppose that $Lf = 0$ in $B_1 \subset \mathbf{R}^d$ and $f \geq 0$ in B_1 then

$$\max_{B_{1/2}} f \leq C_H \min_{B_{1/2}} f.$$

In particular $E_\delta(f) = \{x \in B_{1/2} : |f(x)| < \delta \max_{B_{1/2}} |f|\}$ is empty when δ is sufficiently small.

We will also use the following inequality (equivalence of norms) for any solution f of $Lf = 0$ in B_1 we have

$$\frac{1}{|S_{1/2}|} \int_{S_{1/2}} |f|^2 \leq \max_{B_{1/2}} |f|^2 \leq C \frac{1}{|S_1|} \int_{S_1} |f|^2.$$

Unique continuation property

Definition
A differential operator P is said to have the strong unique continuation property (SUCP) in $\Omega \subset \mathbf{R}^n$ if for any $x \in \Omega$ and any u such that $Pu = 0$ and u vanishes at x of infinite order, $u = 0$ in a neighborhood of x.

Definition
A differential operator P is said to have the weak unique continuation property (WUCP) in a connected open set $\Omega \subset \mathbf{R}^n$ if $Pu = 0$ in Ω and u vanishes at some open subset of Ω implies $u = 0$ in Ω.

Logarithmic convexity: harmonic functions

Let h be a harmonic function in $B_{R_1} \subset \mathbf{R}^d$ and let $0 < R_0 < R < R_1$, denote

$$m(r) = \left(\frac{1}{|B_r|} \int_{B_r} |h|^2 \right)^{1/2}$$

Then $m(R) \leq m(R_0)^\alpha m(R_1)^{1-\alpha}$, where $R = R_0^\alpha R_1^{1-\alpha}$.

In other words the function $F(t) = \log m(e^t)$ is convex.
Exercises: $m(r) = \sum c_k^2 r^{2k}$ and sum of positive log-convex functions is log-convex.

Corollary:

$$\sup_{B_R} |h| \leq C \sup_{B_{R_0}} |h|^\beta \sup_{B_{R_1}} |h|^{1-\beta}.$$

Almgren's frequency function

Let $div(A\nabla f) = 0$ in $B \subset \mathbf{R}^d$. Define $H(r) = \fint_{\partial B_r} |f|^2$.
Then $H'(r) = 2\fint_{\partial B_r} ff_n$.

Almgren's frequency function

$$\mathcal{N}_f(x,r) = \frac{rH'(r)}{H(r)} = \frac{r\fint_{\partial B_r} ff_n}{\fint_{\partial B_r} |f|^2}$$

- If f is a homogeneous polynomial of degree N then $\mathcal{N}_f(0,r) = N$.
- If f vanishes at x with its derivatives up to order N, then $\lim_{r\to 0}\mathcal{N}_f(x,r) = N$

Logarithmic convexity of the norms of elliptic PDE

Theorem (Garofalo-Lin, 1986)

There exist c and r_0 such that $e^{cr}\mathcal{N}_f(x,r)$ is increasing function of r on $(0,r_0)$.

The doubling index of a function is closely connected to its frequency. We define it by

$$N_{2,f}(x,r) = \log \frac{\fint_{\partial B(x,2r)} |f|^2}{\fint_{\partial B(x,r)} |f|^2}$$

Then

$$N_{2,f}(x,r) = \int_r^{2r} \frac{tH_f'(x,t)}{H_f(x,t)} \frac{dt}{t} = c\mathcal{N}_f(x,r_0)$$

for some $r_0 \in (r,2r)$.

Three balls theorem and modified doubling index

The monotonicity theorem and equivalence of norms implies three balls inequality for solutions of elliptic PDEs (Landis 1963):

$$\max_{B_{r_2}} |f| \leq C \max_{B_{r_1}} |f|^{\beta} \max_{B_{r_3}} |f|^{1-\beta},$$

where $0 < r_1 < r_2 < r_3 < R$ and $Lf = 0$ in B_R.

We will use modified doubling index defined by supremum-norms:

$$N_f(x,r) = \log \frac{\max_{B(x,2r} |f|}{\max_{B(x,r)} |f|}, \quad \tilde{N}_f(x,r) = \sup_{2b \subset B(x,2r)} \frac{\max_{2b} |f|}{\max_b |f|}$$

This function is monotone in r and if $\tilde{N}_f(x,r) > N_0$ then $N_f(x,2r) > (1-\epsilon)\tilde{N}_f(x,r)$.

Cauchy uniqueness theorem

Theorem

Suppose that Ω *is a domain with good boundary,* $f \in C^1(\bar{\Omega})$ *and* $Lf = 0$ *in* Ω*. Let* $\Gamma = B \cap \partial\Omega$ *be a non-empty part of the boundary. If* $f|_\Gamma = 0$ *and* $f_n|\Gamma = 0$ *then* $f \equiv 0$*.*

There is also a quantitative version of Cauchy uniqueness

Theorem

Suppose that $Lf = 0$ *in the unit cube and* $f \in C^1(\bar{Q})$*. If* $|\nabla f| \le \varepsilon$ *on one face of the cube and* $|\nabla f| \le 1$ *in* Q*, then*

$$\max_{1/2Q} |\nabla f| \le C\varepsilon^\gamma.$$

Two lemmas of A.Logunov

The two quantitative results on propagation of smallness can be formulated in terms of the frequency function. Let $LF = 0$ in the ball B_R, $R >> 1$

Lemma (Simplex lemma, Logunov, 2016)

Suppose that $\{x_j\} \subset B_1$ *are the vertices of a non-degenerate simplex,* $r < \min_{j \neq k} |x_j - x_k|$ *and* $d > \max |x_j - x_k|$. *Let further* x_0 *be the barycenter of the simplex. There exists* $c > 0$ *and* N_0 *such that if* $N(x_j, r) > N \geq N_0$ *then* $N(x_0, 2d) > (1 + c)N$.

Lemma (Hyperplane lemma, Logunov, 2016)

Suppose that $\{x_j\}_{j=1}^{A^{d-1}}$ *are points ion the* $B_1 \cap \{x_d = 0\}$ *that form a lattice on the hyperplane and* $N(x_j, r) > N$ *for each* j *then* $N(0, 1) > (1 + c)N$.

Distribution of the frequency function

Combining two lemmas above and using simple iteration procedure one can obtain the following statement of the distribution of cubes with large doubling index:

Corollary

Let $Lf = 0$ in CQ and $N = N_f(Q)$, there exists A such that when Q is partitioned into A^d small cubes q the number of cubes with $N_f(q) > N/(1+\epsilon)$ is bounded by A^{d-1-c}.

Quantitative unique continuation

Let $Lf = 0$ in Ω, $|f| \le \varepsilon$ on $E \subset \Omega$, K is a compact subset of Ω then

$$\max_K |f| \le C \sup_\Omega |f|^{1-\alpha} \varepsilon^\alpha.$$

E =Ball, three balls theorems

$|E| > 0$, analytic coefficients, Nadirashvili 1979

$dim(E) > n - 1$, analytic coefficients, E.M. 2004 (capacity)

$|E| > 0$, non-analytic case, Nadirashvili 86, Vessella 2000, E.M. and Vessella 2012:

$\max_K |h| \le C \exp(-c|\log \epsilon|)^\mu) \sup_\Omega |f|, \quad \mu = \mu(n) < 1.$

A new result on quantitative uniqueness, non-analytic coefficients

Theorem (E.M., A. Logunov, 2017)
Let f be a solution of $Lf = 0$. Assume that

$$|f| \le \epsilon \quad \text{on} \quad E \subset \Omega,$$

where $|E| > 0$. Let K be a compact subset of Ω then

$$\max_K |f| \le C \sup_\Omega |f|^{1-\alpha} \epsilon^\alpha,$$

where C, α depend on $L, |E|, \mathrm{dist}(E, \partial\Omega)$ and K (but not on E and f).

Discrete Laplace operator

Discrete Laplace operator on $(hZ)^n$

$$\Delta_h U(x) = h^{-2}(\sum_{j=1}^{n}(U(x+he_j)+u(x-he_j)-2nU(x)).$$

No (naive) unique continuation property.

Logarithmic convexity in Cauchy problem with some boundary data: Falk and Monk 1986, Reinhardt, Han and Háo 1999

Discrete Carleman estimates:
Klibanov and Santosa 1991, Boyer, Hubert and Le Rousseau 2009, Ervedoza and de Gournay 2011.

Logarithmic convexity for discrete harmonic functions

Theorem (M. Guadie, E.M, 2014)

Let Ω be a connected domain in $\mathbf{R}^n$, O be an open subset of Ω, and $K \subset \Omega$ be a compact set. Then there exists C, α and $\delta < 1$ and N_0 large enough such that for any $N \in \mathbf{Z}, N > N_0$ and any discrete harmonic function U on $\Omega \cap (N^{-1}\mathbf{Z})^n$ we have

$$\max_K |U| \leq C(\max_O |U|^\alpha \max_\Omega |U|^{1-\alpha} + \delta^N \max_\Omega |U|).$$

It is clear that on the right-hand side we should have at least $\delta_0^N \max_\Omega |U|$. There is no (weak) unique continuation principle for discrete harmonic functions.

An improvement

Theorem (L. Buhovsky, A. Logunov, E.M., M.Sodin, 2017)
Let $Q_N = [-N, N]^d$, *if* U *is discrete harmonic in* Q_N, $|U| \le 1$ *on* Q_N *and* $|U| \le \varepsilon$ *on some (fixed) portion of* $Q_{N/4}$ *then*

$$\max_{Q_{N/2}} |U| \le C\varepsilon^\alpha + \delta^N.$$

Tool (Discrete version of the Remez inequality)
P is a polynomial of degree n , $P \in \mathbf{R}[x]$ and
$S \subset I \cap \mathbf{Z} = [a, b]$, $|\#S| > 2n$

$$\sup_I |P| \le \left(\frac{8|I|}{|\#S|}\right)^n \sup_E |P|$$