Remez inequality and propagation of smallness for solutions of second order elliptic PDEs

Part III. Eigenfunctions of Laplace-Beltrami operator

Eugenia Malinnikova
NTNU

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Eigenfunctions

Consider a bounded domain in $\mathbf{R}^n$ with Dirichlet boundary conditions or a compact closed manifold. We study the eigenfunctions of the Laplace operator

$$\Delta_M u + \lambda u = 0.$$

Here Δ_M is a uniformly elliptic operator, u is a solution of a second order equation with zero order term.

Wave-scale

Consider an eigenfunction u

$$\Delta_M u + \lambda u = 0,$$

look at the scale $s = c\lambda^{-1/2}$ and do the change of variables $g(x) = u(x_0 + sx)$, then g satisfies an equation

$$Lg + cg = 0$$

with bounded (small) coefficient c and we believe that on this scale g shares properties of the solutions of elliptic equations in divergence form.

Harmonic extension (lifting)

A better way to work on the wave-scale is to introduce a new variable and consider the function

$$h(x,t) = u(x)e^{\sqrt{\lambda}t}.$$

Then

$$\Delta h = 0$$

where Δ is the Laplace-Beltrami operator on $M \times \mathbf{R}$. We have a second order elliptic operator in divergence form and λ is hidden in the behavior of h in the extra direction.

Similar procedure: from spherical harmonics to harmonic functions in $\mathbf{R}^d$.

Application: the density of zeros

Suppose that u is an eigenfunction

$$\Delta_M u + \lambda u = 0$$

and it is positive on some ball B_r. Then h is positive in the cylinder $B_r \times [-r, r]$. By the Harnack inequality the maximum and minimum of h in the smaller cylinder $C_r = B_{r/2} \times [-r/2, r/2]$ are comparable. But

$$\frac{\max_{C_r} h}{\min_{C_r} h} \geq e^{r\sqrt{\lambda}}.$$

It means that $r \leq C_0 \lambda^{-1/2}$. Thus Z_u intersects each ball of radius $c\lambda^{-1/2}$.

Doubling index of eigenfunctions

Theorem (Donnelly-Fefferman, 1988)

Let u be an eigenfunction with eigenvalue λ, then for any cube Q

$$N(u, Q) \leq C\sqrt{\lambda}$$

Idea of the proof Consider $h(x, t)$ then $N(u, B) \leq N(h, B_1)$, where B_1 is a ball containing B. Then we use the almost monotonicity of the doubling index for solutions of elliptic equations and note that if the size B_0 is comparable to M then $N(h, B_0) \leq C\sqrt{\lambda}$.

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In particular the order of vanishing of u is bounded by $c\sqrt{\lambda}$.

Yau's conjecture

Let u be an eigenfunction, $\Delta_M u + \lambda u = 0$, and Z_u be its zero set. Yau conjectured that

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- For $n = 2$ the estimate from below in due to Büning 1978; the best estimate from above is due to Donnelly and Feffreman 1990 $H^1(Z_u) \leq C\lambda^{3/4}$

Some ideas of Donnelly and Fefferman for real analytic case

- For the estimate from below, partition M into cubes with side of the wave length. On each of this cubes $N_u(q) \leq C\sqrt{\lambda}$.
  Claim: At least half of the cubes satisfy $N_u(q) \leq C$ (analytic technique, $\log|u|$)

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  Claim: At least half of the cubes satisfy $N_u(q) \leq C$ (analytic technique, $\log|u|$)
- Estimate from above: take harmonic extension.
  Claim: if h is a harmonic function (in real analytic metric) with $N_h(q) \leq N$ one has $H^d(Z_h) \leq CN$. (intersections with lines and estimates for analytic functions).

New results

- $n = 2$ the estimate $\lambda^{3/4}$ is not sharp, it can be improved.
- (Logunov 2016) there is a polynomial estimate from above in any dimension $H^{d-1}(Z_u) \leq C\lambda^K$ for some $K = K(d, M)$.
- (Logunov 2016) the conjectured estimate from below holds in any dimension $H^{d-1}(Z_u) \geq c\sqrt{\lambda}$.
- for the Dirichlet Laplacian on a subdomain of $\mathbf{R}^d$ with smooth boundary the Yau conjecture holds.

A question of Nadirashvili

Question

Is it true that there exists a constant K_d such that for any harmonic function h in $B_1 \subset \mathbf{R}^d$ such that $h(0) = 0$ the inequality $H^{d-1}(Z_h) \geq K_d$ holds?

This is trivial in dimension two (maximum principle).
There is no "analytic" answer in higher dimensions.

◆

Theorem (Logunov, 2016)

The answer is yes for solutions of elliptic equations in divergence form.

◆

This implies the estimate from below in the Yau's conjecture. Zeros are $c\lambda^{-1/2}$-dense. In each cube on the wave scale the measure of the zero set is at least $K_d \lambda^{-d-1/2}$ and we have $\lambda^{d/2}$ cubes.

Not all doubling indices are large

Suppose that u is a function om a compact manifold, $N_{2,u}(q)$ is the doubling index for the L^2-norm. We partition M into cubes on approximately the same size. Then there are cubes with small doubling index. One may estimate the number of such cubes from the estimates on $\|u\|_\infty/\|u\|_2$.

Now let $Lf = 0$ in CQ, consider the doubling index $\tilde{N}_f$ and a partition of Q into A^d small cubes. Then if $\tilde{N}(q) > N_0$ for each small cube q then $\tilde{N}(Q) > AN_0/2$

Iterating this result we obtain: If $\tilde{N}(Q) > N_0$ and Q is divided into B^d small cubes then for at least half of them $\tilde{N}(q) \leq B^{-\delta}\tilde{N}(Q)$.