Remez inequality and propagation of smallness for solutions of second order elliptic PDEs

Part IV. Smallness propagation from sets of positive measure

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Not all doubling indices are large

Suppose that u is a function om a compact manifold, $N_{2,u}(q)$ is the doubling index for the L^2-norm. We partition M into cubes on approximately the same size. Then there are cubes with small doubling index. One may estimate the number of such cubes from the estimates on $\|u\|_\infty/\|u\|_2$.

Now let $Lf = 0$ in CQ, consider the doubling index $\tilde{N}_f$ and a partition of Q into A^d small cubes. Then if $\tilde{N}(q) > N_0$ for each small cube q then $\tilde{N}(Q) > AN_0/2$

Iterating this result we obtain: If $\tilde{N}(Q) > N_0$ and Q is divided into B^d small cubes then the number of cube where $\tilde{N}(q) \geq \tilde{N}(Q)/2$ is $\leq B^{d-\gamma}$.

A new result on quantitative uniqueness, non-analytic coefficients

Theorem (E.M., A. Logunov, 2017)
Let f be a solution of $Lf = 0$. Assume that

$$|f| \leq \epsilon \quad \text{on} \quad E \subset \Omega,$$

where $|E| > 0$. Let K be a compact subset of Ω then

$$\max_K |f| \leq C \sup_\Omega |f|^{1-\alpha} \epsilon^\alpha,$$

where C, α depend on $L, |E|, \mathrm{dist}(E, \partial\Omega)$ and K (but not on E and f).

Remez inequality for solutions

Let Q be a unit cube. Assume f is a solution to $\mathrm{div}(A\nabla f) = 0$ in C_d and define the doubling index $N = \log \frac{\sup_{2Q}|u|}{\sup_Q |u|}$. Then

$$\sup_Q |u| \leq C \sup_E |u| \left(C \frac{|Q|}{|E|} \right)^{CN}$$

where C depends on A only, E is any subset of Q of a positive measure.

Reformulation

The Remez inequality is equivalent to the following estimate of the sub-level set.

Lemma

Suppose that $\mathrm{div}(A\nabla u) = 0$ *in* $C_d Q$ *and* $\sup_Q |u| = 1$*. Let* $N = N(u, Q) \geq 1$*. Set*

$$E_a = \{x \in Q : |u(x)| < e^{-a}\}.$$

Then

$$|(E_a)| < Ce^{-\beta a/N}|s(Q)|^d,$$

for some $C, \beta > 0$*.*

This lemma implies the propagation of smallness result.

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- Induction step: from $N/2$ and all a to N and all a using induction on a and base 1.

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$$H^{d-1}(\{f = 0\} \cap Q) \leq C_N s(Q)^{d-1}$$

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and if $\{f = 0\} \cap 1/2Q \neq \emptyset$ then

$$H^{d-1}(\{f = 0\} \cap Q) \geq c_N s(Q)^{d-1}.$$

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Then

$$|E_a| \leq Ls(q)^d$$

Number of cubes intersecting the zero set

It remains to estimate L,

$$C_N s(Q)^{d-1} \geq H^{d-1}(Z_f \cap Q) = \sum_{j=1}^{L} H^{d-1}(Z_f \cap q_j) \geq L c_N s(q)^{d-1}$$

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We have also $s(q) = Ce^{-a/N} s(Q)$ and then

$$L s(q)^d \leq C_N (c_N)^{-1} s(q) s(Q)^{d-1} = b_N e^{-a/N} s(Q)^d.$$

Combining with the previous estimate we get the statement of induction base 2.

Choosing the right notation

By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and, in effect, increases the mental power of the race.

Alfred North Whitehead, "'An Introduction to Mathematics"'

Induction step

Let Q_0 be the unit cube in $\mathbf{R}^d$ and let

$$m(f,a) = |\{x \in Q_0 : |u(x)| < e^{-a} \sup_{Q_0} |f|\}|,$$

and

$$M(N,a) = \sup_{*} m(f,a),$$

where the supremum is taken over all elliptic operators $\mathrm{div}(A\nabla\cdot)$ and functions f satisfying the following conditions in $C_d Q_0$:

(i) $A(x) = [a_{ij}(x)]_{1\le i,j\le n}$ is a symmetric uniformly elliptic matrix with Lipschitz entries.

(ii) f is a solution to $\mathrm{div}(A\nabla f) = 0$ in $C_d Q_0$,

(iii) $N(f, Q_0) \le N$.

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Here $a_1 = a - cN\log B$ it appears since $\sup_{Q_0}|f|$ and $\sup_q|f|$ differs, but we have $\sup_q|f| \geq B^{-cN}\sup_Q|f|$.

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$$M(N,a) \le M(N/2, a - cN\log B) + B^{-\gamma}M(N, a - cN\log B)$$

$$\le Ce^{-\beta a/N}\left(e^{-\beta a/N}B^{c\beta} + B^{-\gamma}B^{c\beta}\right)$$