Remez inequality and propagation of smallness for solutions of second order elliptic PDEs
Part V. More results and questions

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March 2018

Two lemmas of A.Logunov

The two quantitative results on propagation of smallness can be formulated in terms of the frequency function. Let $LF = 0$ in the ball B_R, $R >> 1$

Lemma (Simplex lemma, Logunov, 2016)

Suppose that $\{x_j\} \subset B_1$ *are the vertices of a non-degenerate simplex,* $r < \min_{j \neq k} |x_j - x_k|$ *and* $d > \max |x_j - x_k|$*. Let further* x_0 *be the barycenter of the simplex. There exists* $c > 0$ *and* N_0 *such that if* $N(x_j, r) > N \geq N_0$ *then* $N(x_0, 2d) > (1 + c)N$*.*

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Lemma (Hyperplane lemma, Logunov, 2016)

Suppose that $\{x_j\}_{j=1}^{A^{d-1}}$ *are points ion the* $B_1 \cap \{x_d = 0\}$ *that form a lattice on the hyperplane and* $N(x_j, r) > N$ *for each* j *then* $N(0, 1) > (1 + c)N$*.*

Distribution of the frequency function

Combining two lemmas above and using simple iteration procedure one can obtain the following statement of the distribution of cubes with large doubling index:

Corollary

Let $Lf = 0$ *in* CQ *and* $N = N_f(Q)$*, there exists* B *such that when* Q *is partitioned into* B^d *small cubes* q *the number of cubes with* $N_f(q) > N/(1+\epsilon)$ *is bounded by* $B^{d-1-\gamma}$*.*

Propagation of smallness from sets of co-dimension less than one

The assumption that E has positive d-dimensional Lebesgue measure can be relaxed. It is enough to assume that the dimension of E is larger than $d-1$, as in the analytic case. We fix the Hausdorff content of E of some order $d-1+\delta$ with $\delta > 0$ instead.

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Remind that the Hausdorff content of a set $E \subset \mathbf{R}^d$ is

$$C_H^s(E) = \inf\Big\{\sum_j r_j^s : E \subset \cup_j B(x_j, r_j)\Big\},$$

and the Hausdorff dimension of E is defined as

$$\dim_H(E) = \inf\{s : C_H^s(E) = 0\}.$$

Propagation of smallness for the gradients of solutions

let $Lf = 0$ then the inequality

$$\sup_K |\nabla f| \leq C(\sup_E |\nabla f|)^\alpha (\sup_\Omega |\nabla f|)^{1-\alpha}, \qquad (1)$$

holds for all sets E with Hausdorff dimension

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Question: Is it true when $\dim_H(E) > d - 2$?

A related question

Conjecture 1 (Fang-Hua Lin). Let h be a non-zero harmonic function in the unit ball $B_1 \subset \mathbb{R}^d$, $d \geq 3$. Consider

$$N = \log \frac{\sup_{B_1} |\nabla h|}{\sup_{B_{1/2}} |\nabla h|}$$

Is it true that

$$H^{d-2}(\{\nabla h = 0\} \cap B_{1/2}) \leq C_d N^2$$

for some C_d depending only on the dimension?

Cauchy uniquenss from measurable sets on the boundary

Conjecture 2. Assume that u is a harmonic function in the unit ball $B_1 \subset \mathbb{R}^3$ and u is C^∞-smooth in the closed ball $\overline{B_1}$. Let $S \subset \partial B_1$ be any closed set with positive area. Is it true that $\nabla u = 0$ on S implies $\nabla u \equiv 0$?

Remez inequality for eigenfunctions

Let (M, g) be a C^∞ smooth closed Riemannian manifold and let Δ denote the Laplace operator on M. The Remez type inequality for harmonic functions implies the following bound for Laplace eigenfunctions, which was conjectured by Donnelly and Fefferman.

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For any subset E of M with positive volume the following holds:

$$\sup_E |\varphi_\lambda| \geq \frac{1}{C} \sup_M |\varphi_\lambda| \left(\frac{|E|}{C|M|} \right)^{C\sqrt{\lambda}}, \tag{2}$$

where $C = C(M, g) > 1$ does not depend on E and λ.

More about eigenfunctions

Looking at the following example of spherical harmonics $u(x, y, z) = \Re(x + iy)^n$ one can see that L^2 norm of restriction of u on the unit sphere is concentrated near equator very fast and $|u|$ is exponentially small on most of the unit sphere.

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It seems that for negatively curved Riemannian manifolds one can prove better versions of the Remez inequality.

Surfaces of negative curvature

Theorem (Bourgain-Dyatlov, Dyatlov-Jin, 2017)
Under assumption that (M, g) *is a closed Riemannian surface with constant negative curvature the following inequality holds for Laplace eigenfunctions on* M. *Given an open subset* E *of* M *there exists* $c = c(E, M, g) > 0$ *such that*

$$\int_E \varphi_\lambda^2 \geq c \int_M \varphi_\lambda^2.$$

The constant c does not depend on the eigenvalue λ. Note that the situation on closed surfaces of constant negative curvature is different from the case of the sphere.

Traces on curves

A beautiful result by Bourgain and Rudnick states that on a two dimensional torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ equipped with the standard metric the toral Laplace eigenfunctions φ_λ satisfy L^2 lower and upper restriction bounds on curves.

Namely, if S is a smooth curve on T^2 with non-zero curvature and $\lambda > const(S)$, then

$$c\|\varphi_\lambda\|_{L^2(S)} \leq \|\varphi_\lambda\|_{L^2(T^2)} \leq C\|\varphi_\lambda\|_{L^2(S)}.$$

In particular that implies that on a given smooth curve, which is not geodesic, only a finite number of Laplace eigenfunctions can vanish.

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We don't know if this result holds on the sphere.

THANK YOU FOR YOUR ATTENTION !