LIE ALGEBRA OF ISOMETRIES OF A RIEMANNIAN MANIFOLD

Let (M, g) be Riem. mnfd.

Def. A Killing v.f. is v.f. $\xi \in \mathfrak{X}(M)$ s.t. $\mathcal{L}_\xi g = 0$

Let $\xi \in \mathfrak{X}(M)$ and define $A_\xi : TM \to TM$ by

$A_\xi(Y) := -\nabla_Y \xi$

where ∇ is Levi-Civita connection.

LEMMA ξ is Killing iff A_ξ is section of $\mathfrak{so}(TM)$.

$$\Big(g(A_\xi Y, Z) \underset{\text{def.}}{=} -g(\nabla_Y \xi, Z) \underset{\nabla \text{ is torsion-free}}{=}$$

$$= g([\xi, Y], Z) - g(\nabla_\xi Y, Z)$$

$$= \xi(g(Y,Z)) - g(Y, [\xi, Z]) - g(\nabla_\xi Y, Z)$$

$$\underset{\xi \text{ is Killing}}{=} g(Y, \nabla_\xi Z) - g(Y, [\xi, Z])$$

$$\underset{\nabla \text{ is metric}}{=} g(Y, \nabla_Z \xi) = -g(Y, A_\xi Z) \Big)$$

∇ is torsion-free

We want to introduce Killing transport: how to think of Killing v.f. as parallel sections of some bundle.

Consider bundle.

$E = TM \oplus \mathfrak{so}(TM)$

and let us define covariant derivative D on E by

$$D_X(\xi, A) := \left(\nabla_X \xi + A(X),\ \nabla_X A - R(X,\xi) \right)$$

where $R_{XY} = \nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X$ is curvature of ∇.

<u>PROPOSITION</u>: Parallel sections of E w.r.t. D are precisely the Killing vectors. [KOSTANT 1955, GEROCH 1969]

<u>Pf.</u>

$$D_X(\xi, A) = 0 \text{ iff } \begin{cases} A(X) = -\nabla_X \xi \\ \nabla_X A = R(X,\xi) \end{cases}$$ (i.e. $A = A_\xi$ and ξ is Killing)

We need to show that second identity is automatically satisfied. This is "Killing's identity".

$$(\nabla_X A) Y = \nabla_X (AY) - A(\nabla_X Y)$$

$$= -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi$$

$A = A_\xi$

$\Rightarrow$ we take difference

$$(\nabla_X A) Y - (\nabla_Y A) X = -\nabla_X \nabla_Y \xi + \nabla_{\nabla_X Y} \xi$$

$$+ \nabla_Y \nabla_X \xi - \nabla_{\nabla_Y X} \xi$$

∇ is torsion-free $\leftarrow$

$$= -\nabla_X \nabla_Y \xi + \nabla_Y \nabla_X \xi + \nabla_{[X,Y]} \xi$$

$$= R(X,Y)\xi$$

We now use algebraic Bianchi Identity

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

to arrive at

$$\underbrace{(\nabla_X A) Y - R(X,\xi) Y}_{\text{symmetric in } X \leftrightarrow Y} = (\nabla_Y A) X - R(Y,\xi) X$$

On the other hand

$$g(\underbrace{(\nabla_X A)}_{\in \Gamma(\mathfrak{so}(TM))} Y - \underbrace{R(X,\xi)}_{\in \Gamma(\mathfrak{so}(TM))} Y, Z) = -g((\nabla_X A) Z - R(X,\xi) Z, Y)$$

whence $\nabla A - R(-,\xi) \in \left(\mathfrak{so}(T_xM) \otimes T_x^*M\right) \cap T_xM \otimes \bigodot^2 T_x^*M$

at any $x \in M$. This space is easily seen to be zero. $\blacksquare$

$$\Big(\langle f(v)w, z\rangle = \langle f(w)v, z\rangle = -\langle v, f(w)z\rangle$$
$$= -\langle v, f(z)w\rangle = \langle f(z)v, w\rangle$$
$$= \langle f(v)z, w\rangle = -\langle f(v)w, z\rangle \Rightarrow f = 0 \Big)$$

Thus, if M is connected, a Killing v.f. is entirely determined by its value at a point and the value of its first derivative.

PROPOSITION Consider the space of parallel sections of E w.r.t. D. Then Lie bracket inherited from Lie bracket of Killing v.f. is

$$[(\xi, A), (\eta, B)] = (A\eta - B\xi, [A,B] + R(\xi,\eta))$$

Pf. By definition

$$[(\xi,A),(\eta,B)] = (\underbrace{[\xi,\eta]}_{\nabla_\xi\eta - \nabla_\eta\xi = A\eta - B\xi}, -\nabla[\xi,\eta])$$

Now

$$-\nabla_X[\xi,\eta] = -\nabla_X(A_\eta - B_\xi)$$

$$= -(\nabla_X A)_\eta - A(\nabla_X \eta)$$

$$+ (\nabla_X B)_\xi + B(\nabla_X \xi)$$

$$= -R(X,\xi)\eta + AB(X)$$

Killing identity

$$+ R(X,\eta)\xi - BA(X)$$

$$= [A,B](X) + R(\xi,\eta)X$$

(M,g) Riem. mnfd

$G = \left\{ \varphi : M \to M \text{ diffeom.} \mid \varphi^* g = g \right\}$ Lie group of isometries

$\mathfrak{g} = \left\{ \xi \in \mathcal{X}(M) \mid \mathcal{L}_\xi g = 0 \right\}$ Killing algebra (Lie algebra of infin. isometries)

QUESTION: What kind of Lie algebra is $\mathfrak{g}$?

THE FLAT MODEL :

$M = \mathbb{R}^n$

g = Euclidean flat metric

$G = O(n) \ltimes \mathbb{R}^n$

$\mathfrak{g} = euc(n) = so(n) \ltimes \mathbb{R}^n$ Euclidean Lie algebra

isometries fixing the origin — translations

$[A,B] = AB - BA$

$[A,v] = Av$

$[v,w] = 0$

$v, w \in \mathbb{R}^n$

$A, B \in so(n)$

$\underset{0}{euc(n)} = \underset{0}{so(n)} \ltimes \underset{-1}{\mathbb{R}^n}$ is graded Lie algebra

THE ROUND SPHERE

$M = S^n = \{ x \in \mathbb{R}^{n+1} \mid \|x\| = 1 \}$

$G = O(n+1)$ and $\mathfrak{g} = so(n+1)$

Fix $x \in S^n$ and write $V = T_x S^n$. The stabilizer at x

$H \cong O(V)$ with Lie algebra $\mathfrak{h} \cong so(V)$

$\mathfrak{g} \underset{v.s.}{\cong} so(V) \oplus V$

$[A,B] = AB - BA$

$[A,v] = Av$

$[v,w] = \rho(v,w)$

where $\rho = R|_x : \Lambda^2 V \to \mathfrak{so}(V)$ is curvature

$\mathfrak{g}$ is not graded but it is filtered.

Def. A filtration on a Lie algebra is nested sequence of subspaces

$$\mathfrak{g}^\bullet : \cdots \supseteq \mathfrak{g}^{n-1} \supseteq \mathfrak{g}^n \supseteq \mathfrak{g}^{n+1} \supseteq \cdots$$

" degree at least n "

s.t.

$$\bigcap_n \mathfrak{g}^n = 0 \; , \; \bigcup_n \mathfrak{g}^n = \mathfrak{g} \; , \; [\mathfrak{g}^n, \mathfrak{g}^m] \subseteq \mathfrak{g}^{n+m}$$

Def. Let $\mathfrak{g}^\bullet$ be filtration of Lie algebra $\mathfrak{g}$. The associated graded Lie algebra $\mathfrak{g}_\bullet$ is $\mathfrak{g}_\bullet = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n = \mathfrak{g}^n / \mathfrak{g}^{n+1}$.

REM: $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ is well-defined since $[\mathfrak{g}^{n+1}, \mathfrak{g}^m] + [\mathfrak{g}^n, \mathfrak{g}^{m+1}] \subseteq \mathfrak{g}^{n+m+1}$.

We may reverse point of view. Let

$$\mathfrak{a} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{a}_n \qquad [\mathfrak{a}_n, \mathfrak{a}_m] \subseteq \mathfrak{a}_{n+m}$$

be graded Lie algebra. Then a filtered deformation of $\mathfrak{a}$ is a filtered Lie algebra $\mathfrak{g}$ s.t. $\mathfrak{g}_\bullet = \mathfrak{a}$.

Lie brackets of $\mathfrak{g}$ are obtained by adding to those of $\mathfrak{v}$ terms with positive degree.

EX: $\mathfrak{g} = so(n+1)$ is a filtered deformation of $euc(n)$

THM The Lie alg. $\mathfrak{g}$ of isometries of (M^m, g) is filtered and its associated graded Lie algebra is subalgebra of $euc(n)$. "$\mathfrak{g}$ is a filtered subdeformation of $euc(n)$"

Pf

We localize Killing transport at point $x \in M$. Set

$$V = T_x M$$

$$E\big|_x = V \oplus so(V)$$

$\mathfrak{g}$ is subspace of $E\big|_x$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \xrightarrow{ev_x} V' \longrightarrow 0$$

short exact sequence

$$V' = \{ \xi\big|_x \text{ with } \xi \in \mathfrak{g} \} \subseteq V$$

$$\mathfrak{h} = \{ \xi \in \mathfrak{g} \text{ with } \xi\big|_x = 0 \} \cong \{ (0, A) \in \mathfrak{g} \}$$

$$\Longrightarrow \qquad \mathfrak{g} \underset{v.s.}{\cong} \mathfrak{h} \oplus V'$$

Geometrically this amounts to choose splitting $\underset{V'}{\overset{v}{\cap}} \longmapsto \underbrace{(v, X_v)}_{\in \mathfrak{g}}$: for any $v \in V'$ a Killing v.f. ξ s.t. $\xi\big|_x = v$

Tracking back the Lie brackets of $\mathfrak{g}$ on $\mathfrak{h} \oplus V'$ gives

$$[A,B] = AB - BA$$
$$[A,v] = Av + \overbrace{\delta(A,v)}^{\in \mathfrak{h}}$$
$$[v,w] = \underbrace{\alpha(v,w)}_{\in V'} + \underbrace{\rho(v,w)}_{\in \mathfrak{h}}$$

where

$$\alpha(v,w) = X_v w - X_w v$$
$$\delta(A,v) = [A, X_v] - X_{Av}$$
$$\rho(v,w) = [X_v, X_w] - X_{\alpha(v,w)} + R(v,w) \qquad \blacksquare$$

$$\Big([(0,A),(v,X_v)] = (Av, [A,X_v]) = (Av, X_{Av}) + (0, [A,X_v] - X_{Av})$$

Proposition page 2, bracket in $\Gamma_D(E)$ is tensorial

$$[(0,A),(0,B)] = (0,[A,B])$$

$$[(v,X_v),(w,X_w)] = (X_v w - X_w v, [X_v, X_w] + R(v,w))$$
$$= (\alpha(v,w), X_{\alpha(v,w)})$$
$$+ (0, [X_v, X_w] - X_{\alpha(v,w)} + R(v,w)) \Big)$$