RUDIMENTS OF SPINORIAL ALGEBRA & SPIN GEOMETRY

Let (V, η) be v.s. with (pos. def.) inner product η.

Def. The Clifford algebra $Cl(V)$ associated to (V,η) is the algebra generated by V with relation $v^2 = -\eta(v,v)\mathbb{1}$ for all $v \in V \subseteq Cl(V)$.

(More formally it is the quotient of the tensor algebra of V by the two-sided ideal generated by Clifford relations)

EX: Let $\{e_\mu\}$ be orthonormal basis of V. Then

$$e_i e_j + e_j e_i = -2\eta_{ij}\mathbb{1}$$

In particular $e_i e_j + e_j e_i = 0$ if $i \neq j$ and $e_i^2 = -\mathbb{1}$.

We have $Cl(V) \underset{V.S.}{\cong} \Lambda^\bullet V$ but with a different product:

$n = \dim V$	Cl_n
0	$\mathbb{R}$
1	$\mathbb{C}$
2	$\mathbb{H}$
3	$\mathbb{H} \oplus \mathbb{H}$
4	$\mathbb{H}(2)$
5	$\mathbb{C}(4)$
6	$\mathbb{R}(8)$
7	$\mathbb{R}(8) \oplus \mathbb{R}(8)$

Bott periodicity: $Cl_{n+8} \cong Cl_n \otimes \mathbb{R}(16)$

$Cl(V)$ has Lie algebra structure via Clifford commutator and

$$so(V) \cong \left\{ \frac{1}{2} e_i e_j \;\middle|\; i < j \right\}$$

$$e_i \wedge e_j \longmapsto \frac{1}{2} e_i e_j$$

Exponentiating $\mathfrak{so}(V)$ yields the connected component of the spin group

$$\mathrm{Spin}(V) = \left\{ g = v_1 \cdots v_{2n} \in \mathcal{Cl}(V) \mid v_i \in V,\ \eta(v_i, v_i) = +1 \right\}$$

FACT: $g \in \mathrm{Spin}(V)$, $v \in V \Rightarrow g v g^{-1} \in V$

This is the 2-fold cover $\mathrm{Ad} : \mathrm{Spin}(V) \longrightarrow SO(V)$ (ma..
(archetypical example: $\mathrm{Spin}(3) \cong SU(2) \to SO(3) = SO(\mathrm{Im}\,\mathbb{H})$)

Restricting an irreducible module S of $\mathcal{Cl}(V)$ to $\mathrm{Spin}(V)$
we obtain a spinor module.

FACT: $\exists$ (unique up to scaling) inner product $\langle -, - \rangle$ on S

s.t. $\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle \quad \forall v \in V$

$\forall s_1, s_2 \in S$

Let (M^n, g) be orientable Riem. mnfd and

$$SO(M) = \{ u = (e_1, \dots, e_n) \text{ oriented orthonormal basis of } T_xM,\ x \in M \}$$

$$= \{ u : \mathbb{R}^n \to T_xM \text{ orientation-preserving linear isom. s.t. } u^*g = \underbrace{\eta}_{\substack{\text{standard} \\ \text{scalar} \\ \text{product} \\ \text{on } \mathbb{R}^n}},\ x \in M \}$$

the bundle of oriented orthonormal frames.

Def. A <u>spin structure</u> on (M, g) is a principal $Spin(n)$-bundle $Spin(M) \longrightarrow M$ together with bundle morphism

$$\begin{tikzcd}
Spin(M) \arrow[d] \arrow[dr] & \\
SO(M) \arrow[r] & M
\end{tikzcd}$$

which restricts fiberwise to $Ad : Spin(n) \to SO(n)$.

REM: Spin structures do not always exist and even if they do they are not unique (they are parametrized by

$$H^1(M; \mathbb{Z}_2) = Hom(\underbrace{H_1(M)}_{\frac{\pi_1(M)}{[\pi_1(M), \pi_1(M)]}}; \mathbb{Z}_2))$$

Ex: $M = S^n \cong SO(n+1)/SO(n) \cong Spin(n+1)/Spin(n)$

$SO(M) \cong SO(n+1) \to S^n$

$$\begin{tikzcd}
Spin(M) \cong Spin(n+1) \arrow[d, "2:1"] \arrow[dr] & \\
& S^n \\
SO(n+1) \cong SO(M) \arrow[ur] &
\end{tikzcd}$$

$\pi_1(M) = \{1\}$ if $n \geq 2 \Longrightarrow$ unique spin structure.

The associated vector bundle $S(M) = Spin(M) \times_{Spin(n)} S$ is the <u>spinor bundle</u> of (M, g).

SPINOR FIELDS SATISFYING SPECIAL PDES

Def. A spinor field $\varepsilon \in \Gamma(S(M))$ is

- parallel if $\nabla_X \varepsilon = 0 \quad \forall X \in \mathfrak{X}(M)$
- Killing if $\nabla_X \varepsilon = \lambda X \cdot \varepsilon \quad \forall X \in \mathfrak{X}(M)$

$\lambda \longrightarrow$ "Killing constant"

THEOREM If (M, g) has a (non-trivial) parallel spinor then ricci tensor $ric = 0$.

Pf.

$$\nabla_X \varepsilon = 0 \implies R(X,Y)\varepsilon = 0$$

where curvature $R \in \operatorname{End}(S(M)) \otimes \Lambda^2 T^*M$

We take the "Clifford trace" of R

$$0 = 4 \sum_j e_j \cdot R(e_i, e_j)\varepsilon$$

$\longrightarrow R(X,Y,Z,W) := g(R(X,Y)Z, W)$

Now $R(e_i, e_j) = \frac{1}{2} \sum_{k,l} R_{ijkl}\, e_k \wedge e_l$ so acting on spinors

$$R(e_i, e_j)\varepsilon = \frac{1}{4} \sum_{k,l} R_{ijkl}\, e_k \cdot e_l \cdot \varepsilon$$

and hence

$$0 = \sum_{j,k,l} R_{ijkl}\, e_j \cdot e_k \cdot e_l \cdot \varepsilon \qquad (k \neq l \text{ otherwise } R_{ijkl} = 0)$$

$$= \sum_{j,k,l} R_{ijkl} \left(e_{jkl} \underbrace{- \eta_{jk} e_l + \eta_{jl} e_k}_{\text{skew in } k \leftrightarrow l} \right) \cdot \varepsilon$$

$$= \sum_{j,k,l} R_{ijkl} \left(e_{jkl} + 2\eta_{jl} e_k \right) \cdot \varepsilon$$

The first term vanishes by the algebraic Bianchi identity

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0$$

and we are left with

$$0 = -\sum_{j,k,l} R_{jikl}\, 2\eta_{jl}\, e_k \cdot \varepsilon$$

$$= -2\sum_k ric_{ik}\, e_k \cdot \varepsilon$$

Equivalently if we look at the Ricci tensor as endom. of TM we have $ric(X) \cdot \varepsilon = 0 \quad \forall X \in \mathcal{X}(M) \Longrightarrow$

$g(ric(X), ric(X))\, \varepsilon = 0 \Longrightarrow ric(X) = 0$ ∎

Wang '89 <u>Complete, simply connected irreducible mnfds admitting parallel spinors</u>

Holonomy rep.	Geometry	Parallel spinors
$SU(2m+1)$	CY	$(1,1)$
$SU(2m)$	CY	$(2,0)$
$Sp(m)$	HK	$(m+1,0)$
$G_2 \ (\subseteq SO_7)$	exceptional	1
$Spin_7 \ (\subseteq SO_8)$	exceptional	1

THEOREM If (M^n, g) has (non-trivial) Killing spinor with Killing constant $\lambda \in \mathbb{C}$ then $\mathrm{ric} = 4\lambda^2(n-1)g$, i.e., M is Einstein and $\lambda \in \mathbb{R}$ or $\lambda \in i\mathbb{R}$.

(Ricci curvature is bounded below by a positive constant and if (M,g) is complete by a theorem of Myers M is compt)

(If in addition (M,g) is simply connected then by a theorem of Gallot the cone is either flat or irreducible)

Geometry	Cone
Round sphere	Flat
Sasaki-Einstein	CY
3-Sasaki	HK
nearly Kähler	G_2
weak G_2	Spin_7

Let us consider the transpose of Clifford multiplication $V \otimes S \to S$ w.r.t. η and $\langle -,- \rangle$: "Dirac current" $K : S \otimes S \to V$ given by

$$\eta(K(s_1, s_2), v) \overset{(\mathrm{def})}{=} \langle s_1, v \cdot s_2 \rangle$$

REM $\langle -,- \rangle$ and K are Spin-equiv and give corresponding ∇-parallel tensors on TM and $S(M)$

LEMMA Let $\varepsilon_1, \varepsilon_2$ be Killing spinors with the same Killing constant. Then the v.f. $\xi = K(\varepsilon_1, \varepsilon_2)$ is Killing.

Pf.

$$g(\nabla_X \xi, Y) \underset{K \text{ is } \nabla\text{-parallel}}{=} g(K(\nabla_X \varepsilon_1, \varepsilon_2), Y) + g(K(\varepsilon_1, \nabla_X \varepsilon_2), Y)$$

$$= \lambda \langle X \cdot \varepsilon_1, Y \cdot \varepsilon_2 \rangle + \lambda \langle \varepsilon_1, Y \cdot X \cdot \varepsilon_2 \rangle$$

$$= \lambda \left(\langle \varepsilon_1, \underbrace{(Y \cdot X - X \cdot Y)}_{\text{skew in } X \leftrightarrow Y} \cdot \varepsilon_2 \rangle \right) \qquad \blacksquare$$

GEOMETRIC CONSTRUCTION

OF LIE ALGEBRAS (FOF)

(M,g) Riem. spin mnfd

$\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ "Killing superalgebra"

$\mathfrak{k}_{\bar 1} = \left\{ \text{Killing spinors (with } \lambda = \frac{1}{2}) \right\}$

$\mathfrak{k}_{\bar 0} = \left\{ \text{Killing v.f.} \right\}$

QUESTION: $\exists$ natural Lie algebra structure on $\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ compatible with decomposition ?

(i) $[-,-] : \Lambda^2 \mathfrak{k}_{\bar 0} \to \mathfrak{k}_{\bar 0}$ ✓ bracket of v.f.

(ii) $[-,-] : \Lambda^2 \mathfrak{k}_{\bar 1} \to \mathfrak{k}_{\bar 0}$ ✓ Dirac current

(iii) $[-,-] : \mathfrak{k}_{\bar 0} \otimes \mathfrak{k}_{\bar 1} \to \mathfrak{k}_{\bar 1}$ Kosmann "spinorial Lie derivative"

$\xi \in \mathfrak{k}_{\bar 0}, \quad \varepsilon \in \Gamma(S(M))$

Def. $\mathcal{L}_\xi \varepsilon := \nabla_\xi \varepsilon + A_\xi(\varepsilon)$

$\left(\text{if } \mathcal{L}_\xi X = \nabla_\xi X + A_\xi(X) = \nabla_\xi X - \nabla_X \xi = [\xi, X] \right)$

MAIN PROPERTIES OF SPINORIAL LIE DERIVATIVE

$\xi, \eta \in \mathfrak{k}_{\bar{0}}$, $X \in \mathfrak{X}(M)$, $\varepsilon \in \Gamma(S(M))$, $f \in C^\infty(M)$

- $[\mathcal{L}_\xi, \mathcal{L}_\eta]\varepsilon = \mathcal{L}_{[\xi,\eta]}\varepsilon$
- $\mathcal{L}_\xi(X \cdot \varepsilon) = [\xi, X]\cdot\varepsilon + X \cdot \mathcal{L}_\xi \varepsilon$
- $\mathcal{L}_\xi(f\varepsilon) = \xi(f)\varepsilon + f\mathcal{L}_\xi\varepsilon$
- $[\mathcal{L}_\xi, \nabla_X]\varepsilon = \nabla_{[\xi, X]}\varepsilon$

LEMMA If $\xi \in \mathfrak{k}_{\bar{0}}$, $\varepsilon \in \mathfrak{k}_{\bar{1}} \Longrightarrow \mathcal{L}_\xi \varepsilon \in \mathfrak{k}_{\bar{1}}$

Pf.
$$\begin{aligned}
\nabla_X \mathcal{L}_\xi \varepsilon &= \mathcal{L}_\xi \nabla_X \varepsilon - \nabla_{[\xi,X]}\varepsilon \\
&= \lambda \mathcal{L}_\xi(X\cdot\varepsilon) - \lambda[\xi, X]\cdot\varepsilon \\
&= \lambda X \cdot \mathcal{L}_\xi \varepsilon
\end{aligned}$$
$\blacksquare$

Jacobi Identities : $\Lambda^3 \mathfrak{k} \longrightarrow \mathfrak{k}$

$$(x,y,z) \longmapsto [x,[y,z]] - [[x,y],z] - [y,[x,z]]$$

4 components :

1. $\Lambda^3 \mathfrak{k}_0 \to \mathfrak{k}_0$ ✓ Jacobi for v.f.

2. $\Lambda^2 \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1$ ✓ $\mathcal{L}$ is a representation

3. $\mathfrak{k}_0 \otimes \Lambda^2 \mathfrak{k}_1 \to \mathfrak{k}_0$ ✓

X is Killing

$$
\begin{aligned}
g(\mathcal{L}_\xi [\varepsilon_1, \varepsilon_2], X) &= \xi\, g([\varepsilon_1, \varepsilon_2], X) \\
&\quad - g([\varepsilon_1, \varepsilon_2], \mathcal{L}_\xi X) \\
&= \xi \langle \varepsilon_1, X \cdot \varepsilon_2 \rangle - \langle \varepsilon_1, \mathcal{L}_\xi X \cdot \varepsilon_2 \rangle \\
&= \langle \mathcal{L}_\xi \varepsilon_1, X \cdot \varepsilon_2 \rangle + \langle \varepsilon_1, \mathcal{L}_\xi (X \cdot \varepsilon_2) \rangle \\
&\quad - \langle \varepsilon_1, \mathcal{L}_\xi X \cdot \varepsilon_2 \rangle \\
&= \langle \mathcal{L}_\xi \varepsilon_1, X \cdot \varepsilon_2 \rangle + \langle \varepsilon_1, X \cdot \mathcal{L}_\xi \varepsilon_2 \rangle \\
&= g([\mathcal{L}_\xi \varepsilon_1, \varepsilon_2], X) + g([\varepsilon_1, \mathcal{L}_\xi \varepsilon_2], X)
\end{aligned}
$$

$\mathcal{L}_\xi \langle -, - \rangle = 0$

4. $\Lambda^3 \mathfrak{k}_1 \to \mathfrak{k}_1$? but $\mathfrak{k}_0$-invariant

If $(\mathfrak{k}_1 \otimes \Lambda^3 \mathfrak{k}_1^*)^{\mathfrak{k}_0} = 0 \implies \mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$ is a Lie algebra

Normed real division algebras: $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ [HURWITZ]

Hopf fibrations:

$$
\begin{tikzcd}
S^1 \arrow[d, "S^0"] & S^3 \arrow[d, "S^1"] & S^7 \arrow[d, "S^3"] & S^{15} \arrow[d, "S^7"] \\
S^1 & S^2 & S^4 & S^8
\end{tikzcd}
$$

$S^x \subseteq \mathbb{K}^2$

$S^x \cong \mathbb{K}\mathbb{P}^1$

$S^x \subseteq \mathbb{K}$

These are the only examples of fibre bundles where all three spaces are spheres [Adams]

THEOREM (FOF) Applying the Killing superalg. construction to exceptional Hopf fibration, one gets $E_8 = \mathfrak{so}(16) \oplus \mathbb{R}^{128}$

$\mathfrak{so}(9) = \mathfrak{so}(8) \oplus \mathbb{R}^8$

$F_4 = \mathfrak{so}(9) \oplus \mathbb{R}^{16}$

[Killing spinors on S^n $\cong$ parallel spinors on $\mathbb{R}^{n+1} \setminus \{0\}$

In the same $\mathcal{L}_Z \varepsilon = A_Z(\varepsilon)$ and since Z is linear

the endomorphism A_Z is constant]