SUPERGRAVITY

Minkowski spacetime $\mathbb{M}$ =
affine space modeled on v.s. V with Lorentzian inner product
$\eta : V \otimes V \to \mathbb{R}$ (Lorentzian = "$\mathrm{sgn}(\eta) = (1, n-1)$
i.e. mostly minus")

skip

$$\mathfrak{so}(V) = \{ A : V \to V \mid \eta(Av, v) = 0 \ \forall v \in V \}$$

Poincaré algebra $\mathfrak{p} = \mathfrak{so}(V) \ltimes V$ (= Lie algebra of Kill. v.f. of $\mathbb{M}$). It is graded $\underset{0}{\mathfrak{so}(V)} \oplus \underset{-2}{V}$ with

$[A, B] = AB - BA$
$[A, v] = Av$
$[v, w] = 0$
where $A, B \in \mathfrak{so}(V)$, $v, w \in V$.

Case of our interest today : $\dim V = 11$ (let us be concrete)

$\{e_\mu\} = \{e_0, e_1, \ldots, e_{10}\}$ orthonormal basis of V

N.B. : $e_0^2 = -\mathbb{1}$, $e_i^2 = +\mathbb{1}$ if $i \neq 0$ in $\mathcal{C}\ell(V)$

$\mathcal{C}\ell(V) \cong \mathbb{R}(32) \oplus \mathbb{R}(32) \cong \mathrm{End}(S_+) \oplus \mathrm{End}(S_-)$

has two inequiv. irrep. $S_\pm$, distinguished by action of
$\mathrm{vol}|_{S_\pm} = \pm \mathrm{id}$. For us $S = S_-$.

On S we have symplectic form $\langle -,- \rangle$ satisfying

$$\langle v \cdot s_1, s_2 \rangle = - \langle s_1, v \cdot s_2 \rangle \quad \left(= + \langle v \cdot s_2, s_1 \rangle \right)$$

H is invariant under $\mathrm{Spin}^0(V)$:

$$\langle v_1 \cdots v_{2k} \cdot s_1, v_1 \cdots v_{2k} \cdot s_2 \rangle = (-1)^{2k} \langle s_1, v_{2k} \cdots v_1 v_1 \cdots v_{2k} \cdot s_2 \rangle$$

$$= \eta(v_1, v_1) \cdots \eta(v_{2k}, v_{2k}) \langle s_1, s_2 \rangle$$

$$= \pm \langle s_1, s_2 \rangle$$

but for the connected component we get $+1$). Dirac current

$$\eta(K(s_1, s_2), v) = \langle s_1, v \cdot s_2 \rangle$$

is now symmetric.

NOT NOW, LATER

(FACT: $K(s,s)$ is either timelike or lightlike.)

Def. A Lie superalgebra is $\mathbb{Z}_2$-graded v.s. $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with bracket s.t.

(i) $[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subseteq \mathfrak{g}_{\overline{i+j}}$

(ii) $[x,y] = -(-1)^{|x||y|} [y,x]$

(iii) $[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|} [y,[x,z]]$

where $|x| = \begin{cases} 0 & \text{if } x \in \mathfrak{g}_{\bar{0}} \\ 1 & \text{if } y \in \mathfrak{g}_{\bar{1}} \end{cases}$

NOT NOW, LATER

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The Poincaré superalgebra is graded LSA

$$\mathfrak{p} = \underset{0}{\mathfrak{so}(V)} \oplus \underset{-1}{S} \oplus \underset{-2}{V} \quad \left(= \mathfrak{p}_0 \oplus \mathfrak{p}_{-1} \oplus \mathfrak{p}_{-2}\right)$$

$\mathfrak{p}_{\bar{0}} =$ Poincaré algebra $\mathfrak{p}_0 \oplus \mathfrak{p}_{-2}$ $\qquad \mathfrak{p}_{\bar{1}} = S \; \left(= \mathfrak{p}_{-1}\right)$

$[A,B] = AB - BA$

$[A,v] = Av$

$[A,s] = As$ $\qquad A, B \in \mathfrak{so}(V)$

$[v,s] = 0$ $\qquad v, w \in V$

$[s,s] = \kappa(s,s)$ $\qquad s \in S$

$[v,w] = 0$

SKIP ($\mathfrak{so}(V) \subseteq \mathcal{C}\ell(V)$ and As is the restriction to $\mathfrak{so}(V)$ of the spinor rep.)

SKIP (The only non-trivial Jacobi Identity is (DIRAC CURRENT IS SPIN-INVARIANT)

$$[A,[s,s]] \overset{?}{=} [[A,s],s] + [s,[A,s]]$$

Take inner product with $v \in V$:

$$\eta(v, A[s,s]) = -\eta(Av, [s,s]) \overset{\text{by def.}}{=} -\langle s, (Av)\cdot s\rangle$$

$$= -\langle s, A(v\cdot s)\rangle + \langle s, v\cdot(As)\rangle$$

$$= \langle As, v\cdot s\rangle + \langle s, v\cdot(As)\rangle$$

$$= \eta(v, [As,s]) + \eta(v, [s,As])$$

)

Let $\mathcal{P}(V) = \mathrm{Spin}^\circ(V) \ltimes V$ "spin cover of Poincaré group"

Relativistic quantum particles $\longleftrightarrow$ irreducible, unitary reps $\mathcal{H}$ of $\mathcal{P}(V)$

They can be constructed by Wigner "little group" method (special case of Mackey's theory of induced reps)

REM.

Let $P^2 = \sum_{\mu,\nu} \eta^{\mu\nu} e_\mu e_\nu \in \underbrace{\mathcal{U}(\mathfrak{p}_\circ)^{P^\circ}}_{\text{center of univ. envel. algebra of Poincaré algebra}}$

Clearly $[e_\nu, P^2] = 0$

$[so(V), P^2] = 0$ as η is $so(V)$-invariant

Schur Lemma $\longrightarrow$ P^2 acts a scalar multiple of the identity on any irrep. of $\mathcal{P}(V)$

Physically:

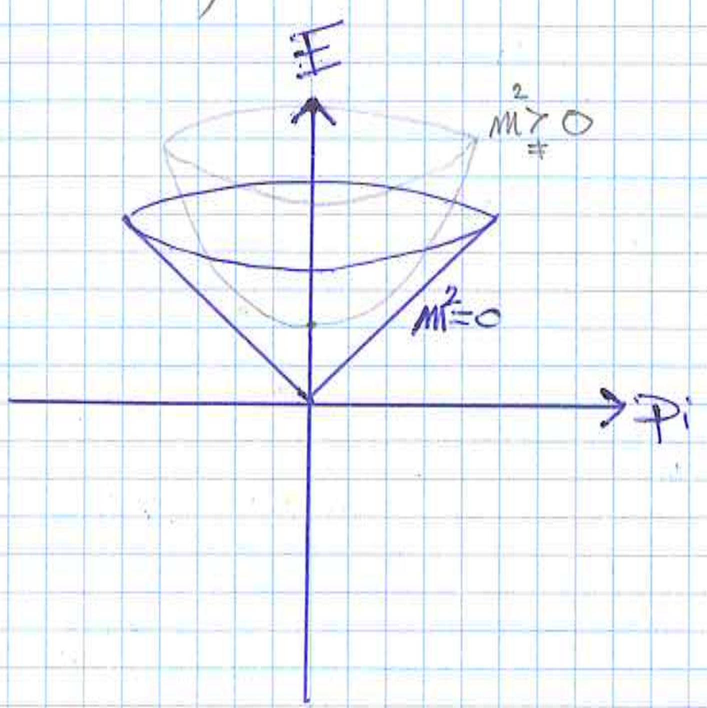
$P^2 = m^2 \, id$ where m is mass of particle and $m^2 > 0$

BASICS ABOUT REPS OF POINCARE' GROUP (in $\dim V = 1,1$)

Since V is abelian and rep. is unitary, $\mathcal{H}$ decomposes as direct sum (really: direct integral) of 1-dim. reps on which V acts by character $\alpha \in V^*$:

$v \longmapsto$ multiplication by $e^{i\alpha(v)}$

The set of characters α which occur are permuted by the action of $Spin^\circ(V)$ on V^* and they form an orbit (since $\mathcal{H}$ is irreducible) ③



Orbits of $Spin^\circ(V)$ on V^*

(Energy and momentum are dual variables to time and space.)

$$\text{Stabilizer } H \cong \begin{cases} Spin(10) & \text{if } m^2 \gneq 0 \\ Spin(9) \ltimes \mathbb{R}^9 & \text{if } m^2 = 0 \end{cases}$$

$$\text{Little group} \overset{(def)}{=} \text{Max. compact subgroup of } H = \begin{cases} Spin(10) & \text{if } m^2 \gneq 0 \\ Spin(9) & \text{if } m^2 = 0 \end{cases}$$

($Spin^\circ(V)$ acts on projectivization of forward light cone, which is S^9, as conformal transformations. The stabilizer is $Spin(9) \ltimes \mathbb{R}^9$)

Consider bundle
$$\begin{array}{c} Spin^\circ(V) \\ \downarrow H \\ \dfrac{Spin^\circ(V)}{H} \end{array}$$
over the orbit

and finite-dim. unitary rep. U of H (such reps. does necessarily factor through little group). Then consider complex Hermitian v.b. over orbit

$$E := Spin^\circ(V) \times_H U$$

and set $\mathcal{H} := L^2(E)$.

We are interested in gravity, so we look for gravitons, ... which are massless.

EX: graviton g $\longleftrightarrow$ $\odot^2_\circ W$ (W spacelike 9-dim.)

(massless) 3-form potential A $\longleftrightarrow$ $\Lambda^3 W$

gravitino ψ $\longleftrightarrow$ $(W \otimes \Sigma)_\circ$.

$(\psi \in \Gamma(S(M) \otimes T^*M))$ 1-form with values in spinor bundle

Σ spinor module of $Spin(9)$

INCOMPLETE HISTORY OF SUPERSYMMETRY

- 1960s: is there a group larger than $P(V)$ whose reps contain Poincaré irreps of different masses and spin?
- 1967: NO (Coleman-Mandula) '72 Wess-Zumino model
- 1975: YES, more or less. (Haag-Łopuszański-Sohnius)

That required the introduction of the Poincaré superalgebra. Irreps of $\mathfrak{p}$ break up into more particles with same mass but different spins

"Supermultiplets" (P^2 is still a Casimir)

'76 d=4 SUGRA (Ferrara, vN, Freedman)

- 1978: $\exists$ irrep. of $\mathfrak{p}$ with field content (g, A, Ψ) (Nahm)

- 1978: Cremmer-Julia-Scherk constructed 11-dimens. supergravity theory predicted by Nahm.

Def. A (bosonic) supergravity background is Lorentzian mnfd (M^{11}, g, F) with closed $F \in \Omega^4(M)$ s.t.

(MAX) $\quad d * F = \frac{1}{2} F \wedge F$

(EIN) $\quad \mathrm{ric}(X, Y) = \frac{1}{2} g(i_X F, i_Y F) - \frac{1}{6} \|F\|^2 g(X, Y)$

Supergravity theories are invariant w.r.t. infinit. transf. depending on spinor fields, e.g., $\delta_\varepsilon \Psi = D \varepsilon + O(\Psi)$

connection

In 11-dim. sugra

$$D_X \varepsilon := \nabla_X \varepsilon - \frac{1}{24}(X \cdot F - 3 F \cdot X) \cdot \varepsilon$$

Def. A spinor field $\varepsilon \in \Gamma(S(M))$ s.t. $D\varepsilon = 0$ is a Killing spinor.

(Let $\mathfrak{k}_{\bar{1}} := \{ \varepsilon \in \Gamma(S(M)) \mid D\varepsilon = 0 \}$, then $\dim \mathfrak{k}_{\bar{1}} \leq 32$.)

EX: $M = \mathbb{M}$, $F = 0$

A Killing spinor is parallel w.r.t. flat metric and there are 32 lin. independent Killing spinors.

There are precisely three other (families of) backgrounds with D flat (Freund & Rubin, Kowalski-Glikman '80s, JOF & Papadopoulos 2003)

1. $AdS_4 \times S^7$ $\quad F \propto dvol(AdS_4)$
2. $AdS_7 \times S^4$ $\quad F \propto dvol(S^4)$
3. a Lorentzian symmetric space KG of solvable Lie group ($\|F\|^2 = 0$)

$$\begin{tikzcd}
 & KG \arrow[d, "ZCL"] & \\
AdS_4 \times S^7 \arrow[ur, "PL"] \arrow[r, "ZCL"'] & \mathbb{M} & AdS_7 \times S^4 \arrow[ul, "PL"'] \arrow[l, "ZCL"]
\end{tikzcd}$$

Penrose showed that near a null geodesic, every Lorentzian manifold looks like a plane wave (= $\exists$ parallel null v.f. BRINKMANN

&

transverse geometry is flat PP-WAVE

&

quasi-recurrent curvature)
$\nabla R = X^\flat \otimes R$!)

THM (FOF, Meessen, Philip 2005)

Every supergravity bkgd has canonically associated a Lie superalgebra $K = K_{\bar{0}} \oplus K_{\bar{1}}$, called the Killing superalgebra.

Pf (sketch)

$$K_{\bar{0}} = \{ \xi \in \mathcal{X}(M) \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0 \}$$

$$K_{\bar{1}} = \{ \varepsilon \in \Gamma(S(M)) \mid D\varepsilon = 0 \}$$ where $D_X \varepsilon = \nabla_X \varepsilon - \frac{1}{24}(X \cdot F - 3F \cdot X) \cdot \varepsilon$

Brackets defined using Dirac current and spinorial Lie derivative.

(i) $[K_{\bar{0}}, K_{\bar{0}}] \subseteq K_{\bar{0}}$ ✓

(ii) $[K_{\bar{0}}, K_{\bar{1}}] \subseteq K_{\bar{1}}$ ✓ (simple exercise using $\mathcal{L}_\xi F = 0$)

(iii) $[K_{\bar{1}}, K_{\bar{1}}] \subseteq K_{\bar{0}}$

One needs to show $K = K(\varepsilon, \varepsilon)$ is Killing v.f. and then use that

$$i_{K(\varepsilon,\varepsilon)} F = -d\omega^{(2)}$$

where $\omega^{(2)}(X,Y) = \langle \varepsilon, (X \wedge Y) \cdot \varepsilon \rangle$

$$\Longrightarrow \mathcal{L}_{K(\varepsilon,\varepsilon)} F = \left(d\, i_{K(\varepsilon,\varepsilon)} + i_{K(\varepsilon,\varepsilon)}\, d \right) F = 0$$

Jacobi Identities:

1. Jacobi for v.f. ✓

2. $\mathcal{L}$ is rep. ✓

3. Dirac current is still $K_{\bar{0}}$-equivariant ✓ $\quad A_K(X) = -\nabla_X K(\varepsilon,\varepsilon) = -2K(\nabla_X \varepsilon, \varepsilon)$

4. $\mathcal{L}_K \varepsilon = \nabla_K \varepsilon + A_K \varepsilon = \frac{1}{24}(K \cdot F - 3F \cdot K) \cdot \varepsilon + A_K \varepsilon = 0$

$\forall \varepsilon \in K_{\bar{1}}$ $(K = K(\varepsilon,\varepsilon))$ algebraic equation which is true ✓

EX:

Bkgd M	$\mathfrak{k} = \mathfrak{k}_{\bar 0} \oplus \mathfrak{k}_{\bar 1}$ $\mathfrak{p}$
$AdS_4 \times S^7$	$\mathfrak{osp}(8\|4)$
$AdS_7 \times S^4$	$\mathfrak{osp}(6,2\|4)$
KG	contraction of $\mathfrak{osp}$

THM If $\dim \mathfrak{k}_{\bar 1} \geq 16 \implies (M, g, F)$ is locally homog.
(FOF, HUSTLER 2012)

Pf. Evaluation $\mathfrak{k}_{\bar 0} \xrightarrow{ev_x} T_x M$ is onto $\forall x \in M$

$$\Updownarrow$$

(M, g, F) locally homog.

CLAIM: $[\mathfrak{k}_{\bar 1}, \mathfrak{k}_{\bar 1}] \xrightarrow{ev_x} T_x M$ is onto

In other words if $S' \subseteq S$ with $\dim S' \geq 16$ then $[S', S'] = V$.
If this is not the case $\exists \underset{\neq 0}{v} \in V$ s.t. $v \perp [S', S']$ i.e.

$$0 = \eta(v, [s_1, s_2]) = \langle s_1, v \cdot s_2 \rangle \quad \forall s_1, s_2 \in S'$$

$$\implies v\cdot : S' \longrightarrow (S')^\perp$$

Now $\dim (S')^\perp < 16$ hence $v\cdot$ has a nonzero kernel and v is lightlike ($v^2 = -\eta(v,v) \mathbb{1}$). Hence $[S', S']^\perp$ is v.s. consisting of lightlike vectors

$$\implies \dim [S', S']^\perp = 1 \quad \text{i.e.} \quad [S', S']^\perp = \mathbb{R} v$$

Now $[S', S'] = (\mathbb{R}v)^{\perp} = \mathbb{R}v \overset{\perp}{\oplus} W$ ⑥

$\hookrightarrow$ spacelike 9-dimens.

FACT: Dirac current $[s,s]$ is causal (lightlike or timelike) $\forall s \in S$

$\Rightarrow$ $\forall s \in S'$ $[s,s] \in \mathbb{R}v$ and by polarization

$[S', S'] \subseteq \mathbb{R}v$, a contradiction ∎