RECONSTRUCTION OF HIGHLY SUPERSYMMETRIC BKGDS

①

BEFORE

Def A supergravity background is Lorentzian mnfld (M^{11}, g, F) $\quad F \in \Omega^4(M) \quad dF = 0$

(MAX) $\quad d\star F = \frac{1}{2} F \wedge F$

(EIN) $\quad ric(X,Y) = \frac{1}{2} g(i_X F, i_Y F) - \frac{1}{6} \|F\|^2 g(X,Y)$

SKIP

$\mathfrak{K} = \mathfrak{K}_{\bar 0} \oplus \mathfrak{K}_{\bar 1}$ Killing superalgebra

$$\mathfrak{K}_{\bar 0} = \left\{ \xi \in \mathfrak{X}(M) \mid \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0 \right\}$$

$$\mathfrak{K}_{\bar 1} = \left\{ \varepsilon \in \Gamma(S(M)) \mid D\varepsilon = 0 \right\}$$

$$D_X \varepsilon := \nabla_X \varepsilon - \frac{1}{24}(X \cdot F - 3F \cdot X) \cdot \varepsilon$$

Def (M, g, F) is

(i) supersymm. if $\dim \mathfrak{K}_{\bar 1} \geqslant 1$

(ii) highly supersymm. if $\dim \mathfrak{K}_{\bar 1} \underset{\neq}{\geq} 16 \left(= \frac{\dim S}{2} \right)$

THM I $\mathfrak{K}$ is a filtered subdef. of Poincaré superalgebra $\mathfrak{p}$. (HSF, δ, −)

Pf. "Killing supertransport"

$\mathbb{E} = \mathbb{E}_{\bar 0} \oplus \mathbb{E}_{\bar 1} \qquad \mathbb{E}_{\bar 0} = TM \oplus \mathfrak{so}(TM)$

$\qquad\qquad\qquad\qquad \mathbb{E}_{\bar 1} = S(M)$

Define connection on E

$$D_X \begin{pmatrix} \xi \\ A \\ \varepsilon \end{pmatrix} := \begin{pmatrix} \nabla_X \xi + A(X) \\ \nabla_X A - R(X,\xi) \\ D_X \varepsilon \end{pmatrix}$$

so that $K \cong \{ D\text{-parallel sections of } E \}$. We then localize at $x \in M$:

$$V = T_x M \qquad \Longrightarrow \qquad E\big|_x = so(V) \oplus S \oplus V$$

$$S = S(M)\big|_x$$

and K is a subspace of $E|$,

We have short exact sequence ~~as~~ in classical case and by choosing a splitting $X : V' \to so(V)$ we have

$$K \underset{v.s.}{\cong} h \oplus S' \oplus V'$$

$$h = \{ \xi \in K_{\bar 0} \text{ with } \xi\big|_x = 0 \} \subseteq so(V)$$

$$S' = \{ \varepsilon\big|_x \text{ where } \varepsilon \in K_{\bar 1} \} \subseteq S$$

$$V' = \{ \xi\big|_x \text{ where } \xi \in K_{\bar 0} \} \subseteq V$$

Tracking back Lie brackets of K on $h \oplus S' \oplus V'$ give

$$\left.\begin{array}{l}
[A,B]=AB-BA \\
[A,v]=Av+\overbrace{\delta(A,v)}^{\in\mathfrak{h}} \\
[A,s]=As \\
[v,s]=\overbrace{\beta(v,s)}^{\in S'} \\
[s,s]=\kappa(s,s)+\overbrace{\gamma(s,s)}^{\in\mathfrak{h}} \\
[v,w]=\underbrace{\alpha(v,w)}_{\in V'}+\underbrace{\rho(v,w)}_{\in\mathfrak{h}} \\
A,B\in\mathfrak{h} \\
v,w\in V' \\
s\in S'
\end{array}\right\} \text{FILT. SUBDEF. OF } \mathfrak{p}$$

where α, δ, ρ are as in classical case

$$\beta(v,s)=\beta^{\varphi}(v,s)+X_v(s)$$

$$\gamma(s,s)=\gamma^{\varphi}(s,s)-X_{\kappa(s,s)}$$

and if $\varphi=F|_x\in\Lambda^4V^*$

$$\beta^{\varphi}(v,s):=\frac{1}{24}(v\cdot\varphi-3\varphi\cdot v)\cdot s$$

$$\gamma^{\varphi}(s,s)(v):=-2\kappa(\beta^{\varphi}(v,s),s) \qquad \blacksquare$$

Applications :

THM II Let (M^{11}, g, F) be Lorentzian spin manifd with (T_oF, δ_F) closed $F \in \Omega^4(M)$. If $\dim K_{\bar{1}} \geq 16$ then (EIN) & (MAX) are automatically satisfied.

Sketch of proof

Let $\varepsilon \in \Gamma(S(M))$ and define forms

$$\omega^{(1)}(X) = \langle \varepsilon, X \cdot \varepsilon \rangle$$

$$\omega^{(2)}(X_1, X_2) = \langle \varepsilon, (X_1 \wedge X_2) \cdot \varepsilon \rangle$$

$$\omega^{(5)}(X_1, \dots, X_5) = \langle \varepsilon, (X_1 \wedge \cdots \wedge X_5) \cdot \varepsilon \rangle$$

NON-TRIVIAL FACTS: If $D\varepsilon = 0$ then $\left(K = K(\varepsilon, \varepsilon) \right)$

(i) $d\omega^{(2)} = -i_K F$

(ii) $d\omega^{(5)} = i_K {*F} - \omega^{(2)} \wedge F$

Let us compute

$$0 = *\mathcal{L}_K F = \mathcal{L}_K *F = d i_K *F + i_K d*F$$

$$\underset{\underset{(ii)}{\downarrow}}{=} d(\omega^{(2)} \wedge F) + i_K d*F = d\omega^{(2)} \wedge F + i_K d*F$$

$$\underset{\underset{(i)}{\downarrow}}{=} -\frac{1}{2} i_K (F \wedge F) + i_K d*F = i_K \left(d*F - \frac{1}{2} F \wedge F \right)$$

$\Longrightarrow$ if $\dim K_{\bar{1}} \geq 16$ then (MAX) holds.

Now the Jacobi Identity $[S'S'V]$ in $\mathfrak{k}=\mathfrak{k}_{\bar 0}\oplus\mathfrak{k}_{\bar 1}$ gives ③

$$\frac{1}{2}R(v,\kappa(s,s))w = \kappa((X_v\beta)(w,s),s)$$
$$-\kappa(\beta_v(s),\beta_w(s))$$
$$-\kappa(\beta_w\beta_v(s),s)$$

for all $s\in S'$, $v,w\in V$

$$\Big\Downarrow$$ taking trace and using plenty of Clifford technology...

$$\mathrm{Ric}(v,\kappa(s,s)) = -\frac{1}{2}F^2_{ab}v^a\langle e^b\cdot s,s\rangle + \frac{1}{4}\|F\|^2\langle v\cdot s,s\rangle$$
$$+\frac{1}{6}\Big\langle\Big(\underbrace{\overset{=0\text{ by (MAX)}}{\cancel{v\wedge F\wedge F+2\iota_v\delta F}}}_{\in\Lambda^2V}\underbrace{-v\wedge\overset{=0}{\cancel{dF}}}_{\in\Lambda^5V}\Big)\cdot s,s\Big\rangle$$

$\odot^2S\cong\Lambda^1V\oplus\Lambda^2V\oplus\Lambda^5V$ but here we only have S'!

It remains only Λ^1V and (EW) holds. ▨

Let $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be filt. subdef. of $\mathfrak{p}$ with components β, γ as in proof of THM I (i.e. $\beta=\beta^{\varphi}+X$, $\gamma=\gamma^{\varphi}-X_{\kappa}$ for some $X: V' \to \mathfrak{so}(V)$ and closed $\varphi\in\Lambda^4 V^*$).

Then we say that $\mathfrak{g}$ is realizable.

THM III

(FOF, S.-) Let $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ be realizable filt. subdef. of $\mathfrak{p}$ with $\dim\mathfrak{g}_{\bar 1}\gneqq 16$. Then $\exists$ supergravity bkgd (M,g,F) s.t. $\mathfrak{g}$ is (a subalgebra of) its Killing superalgebra.

(RECONSTRUCTION OF HIGHLY SUPERSYMMETRIC BACKGROUNDS).

THM IV

(FOF, S.-) The classification of max. supersymm. bkgds is easily recovered classifying (realizable) filt. subdef. $\mathfrak{g}=\mathfrak{g}_{\bar 0}\oplus\mathfrak{g}_{\bar 1}$ of $\mathfrak{p}$ with $\dim\mathfrak{g}_{\bar 1}=32$.

OUTLOOK: Systematic way to search for highly supersymm. bkgds.

KILLING SPINOR EQS

FROM COHOMOLOGY

(Systematic approach to determining which spinor eqs give rise to a Lie superalgebra)

Filt. def. are governed by Spencer cohomology, a bi-graded refinement of Chevalley–Eilenberg cohomology. Let us see it for Poincaré superalgebra $\mathfrak{p} = \underset{0}{\mathfrak{so}(V)} \oplus \underbrace{\underset{-1}{S} \oplus \underset{-2}{V}}_{\mathfrak{p}_-}$ in any dimension.

$$C^q = \mathfrak{p} \otimes \Lambda^q(\mathfrak{p}_-^*)$$

$$C^q = \bigoplus_{p\in\mathbb{Z}} C^{p,q} \qquad (\deg(\mathfrak{p}_i^*) = -i)$$

$$\partial : C^q \to C^{q+1} \; , \; \partial^2 = 0 \; , \; \deg(\partial) = 0$$

$$\longrightarrow H^q = \bigoplus_{p\in\mathbb{Z}} H^{p,q}$$

We are interested in $q=2$ and $p=2,4$

$$\begin{tikzcd} C^{2,1} \arrow[r, "\partial"] & C^{2,2} \arrow[r, "\partial"] & C^{2,3} \\ V \to \mathfrak{so}(V) & \begin{array}{c} \Lambda^2 V \to V \\ V\otimes S \to S \\ \odot^2 S \to \mathfrak{so}(V) \end{array} & \cdots \end{tikzcd}$$

$$\begin{tikzcd} C^{4,2} \arrow[r, "\partial"] & C^{4,3} \\ \Lambda^2 V \to \mathfrak{so}(V) & \cdots \end{tikzcd}$$

$H^{1,2}$ ⇒ always 1-form (for any $d = \dim V$)

$$\big\downarrow$$

d=11. $H^{2,2} \cong \Lambda^4 V$ as rep. of $\mathfrak{so}(V)$

Component $\beta : V \otimes S \to S$ of canonical Spencer cocycle is

$$\beta(v,s) = \frac{1}{24}(v\cdot\varphi - 3\varphi\cdot v)\cdot s$$

$$\big\uparrow \text{ gravitino connection}$$

d=4 $\quad H^{2,2} \cong \underbrace{\underset{\underset{A}{\ni}}{\Lambda^0 V} \oplus \underset{\underset{C}{\ni}}{\Lambda^1 V} \oplus \underset{\underset{B}{\ni}}{\Lambda^4 V}}$

auxiliary fields in "old" off-shell formulation of d=4 N=1 supergravity

gravitino connection $\big\downarrow$

$$\beta(v,s) = v\cdot(A+B)\cdot s - C\cdot v\cdot \mathrm{vol}\cdot s + \eta(C,v)\,\mathrm{vol}\cdot s$$

The Killing superalgebra exists without any differ. constraints on (A,B,

$$\begin{array}{lllll} (\text{Mink}, & AdS_4, & AdS_3\times\mathbb{R}, & \mathbb{R}\times S^3, & \text{Nappi-Witten} \\ & C=0 & C \text{ spacelike} & C \text{ timelike} & C \text{ lightlike} \end{array}$$