On a class of 3-algebras

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Plan of the talk:

- Jordan algebras and Jordan triple systems
- Triple systems in Geometry and Supergravity
- Kantor triple systems
- TKK construction
- Infinite-dimensional and finite-dimensional Kantor triple systems
- Classification of Kantor triple systems
- Exceptional Kantor triple systems

Jordan algebras

Def. A *Jordan algebra* is a commutative algebra A satisfying

$$a \circ (b \circ a^2) = (a \circ b) \circ a^2$$

for all $a, b \in A$.

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Jordan, von Neumann and Wigner classified in '34 finite-dimensional simple formally real Jordan algebras:

- the algebras $\mathfrak{h}_n^{\mathbb{R}}$, $\mathfrak{h}_n^{\mathbb{C}}$, $\mathfrak{h}_n^{\mathbb{H}}$ of Hermitian real, complex and quaternionic $n \times n$ matrices with multiplication

$$a \circ b = \frac{1}{2}(ab + ba)$$

- the spin factors $J(\mathbb{R}^n) = \mathbb{R}^n \oplus \mathbb{R}$ with multiplication

$$(v, \alpha) \circ (w, \beta) = (\alpha w + \beta v, \langle v, w \rangle + \alpha\beta)$$

- the *Albert algebra* $\mathfrak{h}_3^{\mathbb{O}}$ of 3×3 Hermitian matrices over octonions

Jordan triple systems

Given any Jordan algebra $(A, \circ)$ one can define a triple product on A by

$$(xyz) := (x \circ y) \circ z + (z \circ y) \circ x - (x \circ z) \circ y$$

where $x, y, z \in A$. It holds:

1. $(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz))$
2. $(xzy) = (yzx)$

for all u, v, x, y, z.

Def. [Meyberg '70] A *Jordan triple system* (shortly, JTS) is a v.s. with a triple product satisfying 1. and 2. above.

Examples of JTS.

- the space $\mathrm{Mat}(p,q;\mathbb{K})$ of *$p \times q$ matrices* over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with

$$(xyz) = xy^*z + zy^*x$$

  and subalgebras (e.g. Hermitian matrices $\mathfrak{h}_p^{\mathbb{K}}$ when $p = q$)
- the *spin factors* $J(\mathbb{R}^n)$ with the triple product naturally induced by Jordan multiplication
- the *Albert algebra* $\mathfrak{h}_3^{\mathbb{O}}$ with the induced triple product and its subalgebra $M(1,2;\mathbb{O})$

The classification of finite-dimensional simple real JTS is due to Loos in '77 and Neher in '80.

Triple systems in Geometry and Supergravity

- Jordan algebras and Jordan triple systems are used in theory of *Hermitian symmetric spaces* [Satake, Bertram] and in *CR geometry* [Kaup, Zaitsev]
- Triple systems satisfying similar (but different) axioms are closely related to *quaternionic Kähler symmetric spaces* [Wolf '65; Alekseevsky, Cortés '05]
- Jordan algebras were used to construct $N = 2$ Maxwell-Einstein $D = 5$ *supergravity* theories [Günaydin, Sierra, Townsend '84]
- $N = 5$ 3-*algebras* appear in $D = 3$ superconformal Chern-Simons theories describing dynamics of branes in supergravity [Aharony, Bergman, Jafferis' 08], [de Medeiros, Figueroa-O'Farrill, Méndez-Escobar, Ritter '09]

Kantor triple systems

I. L. Kantor introduced in '72 a generalization of Jordan triple systems.

Def. A *Kantor triple system* (shortly, KTS) is a v.s. V with a triple product satisfying

1. $(uv(xyz)) = ((uvx)yz) - (x(vuy)z) + (xy(uvz))$
2. $K_{K_{uv}(x)y} = K_{(yxu)v} - K_{(yxv)u}$

for all $u, v, x, y, z \in V$. The "Kantor tensor"

$$K_{xy}(z) = (xzy) - (yzx)$$

measures the failure of the product to be symmetric in outer arguments. If $K_{xy} = 0$ for all $x, y \in V$ then one gets a JTS.

Examples of KTS.

- The space $\mathrm{Mat}(p,q;\mathbb{K})$ of $p\times q$ matrices over $\mathbb{K}=\mathbb{R},\mathbb{C},\mathbb{H}$ with

$$(xyz)=xy^*z+zy^*x-zx^*y$$

- The space $\mathrm{Mat}(p,q;\mathbb{K})\times\mathrm{Mat}(q,r;\mathbb{K})$ with mixed triple product

$$\left(\begin{pmatrix}x_1\\x_2\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\begin{pmatrix}z_1\\z_2\end{pmatrix}\right)=\begin{pmatrix}x_1y_1^*z_1+z_1y_1^*x_1-z_1x_2y_2^*\\x_2y_2^*z_2+z_2y_2^*x_2-y_1^*x_1z_2\end{pmatrix}$$

where $x=(x_1,x_2)$, $y=(y_1,y_2)$ and $z=(z_1,z_2)$.

Simple real KTS of *classical* type that are not JTS are given by (modifications of) these examples [Asano, Kaneyuki '91]. Some examples of exceptional type have been considered by Mondoc in '07.

Construction of Kantor triple systems

Let $\mathfrak{g} = \mathfrak{g}_{-d} \bigoplus \cdots \bigoplus \mathfrak{g}_{+d}$ be a Lie algebra with a *$\mathbb{Z}$-grading* with equal depth and height. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a *grade-reversing involution* of $\mathfrak{g}$. Set $V = \mathfrak{g}_{-1}$ and

$$(xyz) := [[x, \sigma(y)], z]$$

for all $x, y, z \in V$.

Construction of Kantor triple systems

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$$(xyz) := [[x, \sigma(y)], z]$$

for all $x, y, z \in V$. Then

$$\begin{aligned}
(uv(xyz)) &= [[u, \sigma(v)], [[x, \sigma(y)], z]] \\
&= [[[[u, \sigma(v)], x], \sigma(y)], z] + [[x, [[u, \sigma(v)], \sigma(y)]], z] + [[x, \sigma(y)], [[u, \sigma(v)], z]] \\
&= ((uvx)yz) + [[x, \sigma[[\sigma(u), v], y]], z] + (xy(uvz)) \\
&= ((uvx)yz) - (x(vuy)z) + (xy(uvz)) ,
\end{aligned}$$

for all $u, v, x, y, z \in V$. In other words the *principal identity* of KTS holds.

Construction of Kantor triple systems

Let $\mathfrak{g} = \mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{+d}$ be a Lie algebra with a *$\mathbb{Z}$-grading* with equal depth and height. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a *grade-reversing involution* of $\mathfrak{g}$. Set $V = \mathfrak{g}_{-1}$ and

$$(xyz) := [[x, \sigma(y)], z]$$

for all $x, y, z \in V$.

- If *depth $d = 1$* then

$$\begin{aligned} K_{xy}(z) &= (xzy) - (yzx) = [[x, \sigma(z)], y] - [[y, \sigma(z)], x] \\ &= [[x, y], \sigma(z)] = 0 \end{aligned}$$

that is the Kantor tensors all vanish. This is the case of *JTS*.

- *KTS* correspond to *depth $d = 2$*.

TKK construction revisited

Let V be a centerless KTS and define the operators, for $x, y, z \in V$:

$$L_{xy}(z) := (xyz)\,, \qquad \varphi_x(z) := L_{zx}\,, \qquad D_{xy}(z) := -\varphi_{K_{xy}(z)}\,.$$

$$\begin{array}{ccccccccccc} \mathfrak{g} = \mathfrak{g}(V) = \langle K_{xy}\rangle & \oplus & V & \oplus & \langle L_{xy}\rangle & \oplus & \langle \varphi_x\rangle & \oplus & \langle D_{xy}\rangle \\ -2 & & -1 & & 0 & & 1 & & 2 \end{array}$$

The Lie bracket is defined by

$$[x, y] := K_{xy}\,, \qquad [A, z] := A(z)$$

for A either L_{xy}, φ_x or D_{xy} and extended using

- transitivity (if $A \in \mathfrak{g}_i$, $i \geqslant 0$, and $[A, z] = 0$ for all $z \in \mathfrak{g}_{-1}$ then $A = 0$)
- Jacobi identity, since $\mathfrak{g}$ is fundamental ($\mathfrak{g}_{-1}$ generates $\mathfrak{g}_{-2}$).

TKK construction revisited

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$$\begin{array}{cccccccccc} \mathfrak{g} = \mathfrak{g}(V) = \langle K_{xy} \rangle & \oplus & V & \oplus & \langle L_{xy} \rangle & \oplus & \langle \varphi_x \rangle & \oplus & \langle D_{xy} \rangle \\ -2 & & -1 & & 0 & & 1 & & 2 \end{array}$$

- $\mathfrak{g}(V)$ is a subalgebra of the Tanaka prolongation $\mathfrak{g}^\infty$ of $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$;
- The following map is a grade-reversing involution of $\mathfrak{g}(V)$

$$\sigma : \qquad K_{xy} \longleftrightarrow D_{xy}\,, \qquad x \longleftrightarrow -\varphi_x\,, \qquad L_{xy} \longleftrightarrow -L_{yx}$$

TKK construction revisited

Thm I. There is *one-to-one correspondence* between *simple KTS V* and *pairs $(\mathfrak{g}, \sigma)$*, where $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \cdots \oplus \mathfrak{g}_2$ is simple Lie algebra with a 5-grading and σ a grade-reversing involution. Furthermore $\mathfrak{g}$ is finite-dimensional (resp. linearly compact) if and only if V is finite-dimensional (resp. linearly compact).

Classification of Kantor triple systems

We classified the *simple KTS over* $\mathbb{C}$. Let us start with linearly compact case. By a well-known theorem of E. Cartan, simple *infinite-dimensional Lie algebras of vector fields* on an m-dimensional manifold are (locally, formally) isomorphic to:

- $W_m = \left\{ \sum_{i=1}^m P_i \frac{\partial}{\partial x_i} \mid P_i \in \mathbb{C}[[x_1, \ldots, x_m]] \right\}$
- $S_m = \left\{ X \in W_m \mid \mathrm{div}(X) = 0 \right\}$
- $H_m = \left\{ X \in W_m \mid \mathcal{L}_X \omega = 0 \right\}$ $(m = 2k)$, where $\omega = \sum_{i=1}^k dx_i \wedge dx_{k+i}$ is symplectic form
- $K_m = \left\{ X \in W_m \mid \mathcal{L}_X \alpha = f\alpha \text{ for } f \in \mathbb{C}[[x_1, \ldots, x_m]] \right\}$ $(m = 2k+1)$ where $\alpha = dx_m + \sum_{i=1}^k x_i dx_{k+i}$ is contact form

Infinite-dimensional KTS

Thm II. There are *no infinite-dimensional simple* linearly compact KTS.

Proof. Any $\mathbb{Z}$-grading of W_m is determined by assigning degrees

$$\deg(x_i) = k_i \ , \qquad \deg(\frac{\partial}{\partial x_i}) = -k_i \ ,$$

for a m-tuple of integers $(k_1, \ldots, k_m)$. In particular

$$\deg(x_i^n \frac{\partial}{\partial x_j}) = nk_i - k_j$$

and W_m does not admit any 5-grading ($\mathfrak{g}_i \neq 0$ for infinitely many i). Similarly for S_m, H_m and K_m. ■

Structure theory for finite-dimensional KTS

Problem: How to classify the grade-reversing involutions of finite-dimensional simple 5-graded Lie algebras?

Thm III. There is a bijection between isomorphism classes of $(\mathfrak{g}, \sigma)$ and isomorphism classes of $(\mathfrak{g}^o, \theta)$, where $\mathfrak{g}$ is a simple complex $\mathbb{Z}$-graded Lie algebra with a grade-reversing involution σ and $\mathfrak{g}^o$ is a real absolutely simple $\mathbb{Z}$-graded Lie algebra with a grade-reversing Cartan involution.

Solution:

Finite dimensional simple KTS over $\mathbb{C}$ are in one-to-one correspondence with 5-gradings of real forms of simple complex Lie algebras.

The algebra of derivations

Thm IV. Let V be a simple KTS with associated real form $(\mathfrak{g}^o, \theta)$. Let $(\mathfrak{g}^o_0)^{ss}$ be the semisimple part of $\mathfrak{g}^o_0$ and $\mathfrak{g}^o_0 = \mathfrak{k} \oplus \mathfrak{p}$ its Cartan decomposition. Then the *Lie algebra of derivations* of V is

$$\mathfrak{der}(V) = \mathfrak{k} \otimes \mathbb{C}\ .$$

Proof. The space $\mathfrak{der}(V)$ consists of (the restriction to $V = \mathfrak{g}_{-1}$ of) the 0-degree derivations of $\mathfrak{g}$ commuting with σ. Any derivation is inner and it is known that the restriction to $(\mathfrak{g}^o_0)^{ss}$ of the Cartan involution of $\mathfrak{g}^o$ is still a Cartan involution. ■

Finite-dimensional KTS

Thm V. Up to isomorphism there are 8 infinite series of classical simple KTS and 23 exceptional cases.

The exceptional KTS can be divided into three main classes, depending on the grading of the associated Lie algebra:

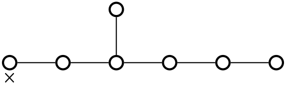
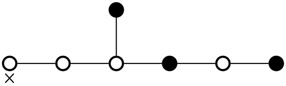
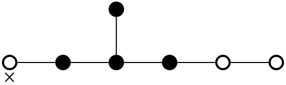
(i) of *contact type* if $\dim \mathfrak{g}_{-2} = 1$;

(ii) of *extended Poincaré type* if $\mathfrak{g}_{-2} = U$ and $\mathfrak{g}_0 \supset \mathfrak{so}(U)$;

(iii) of *special type* otherwise.

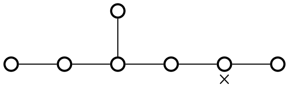
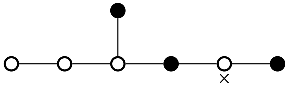
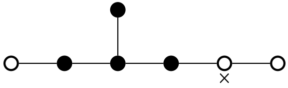
Exceptional KTS associated to E_7

Contact type

$\mathfrak{g}^o$	Satake Diagram	KTS
EV		$V = \mathbb{S}^+$ $\mathfrak{der}(V) = \mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{so}(6,\mathbb{C})$
EVI		$V = \mathbb{S}^+$ $\mathfrak{der}(V) = \mathfrak{gl}(6,\mathbb{C})$
$EVII$		$V = \mathbb{S}^+$ $\mathfrak{der}(V) = \mathfrak{so}(2,\mathbb{C}) \oplus \mathfrak{so}(10,\mathbb{C})$

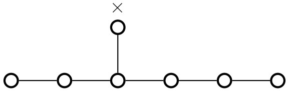
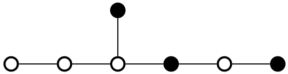
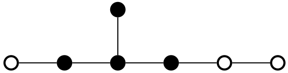
Exceptional KTS associated to E_7

Poincaré type

$\mathfrak{g}^o$	Satake Diagram	KTS
$E\,V$		$V = \mathbb{S}^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(5,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$
$E\,VI$		$V = \mathbb{S}^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(3,\mathbb{C}) \oplus \mathfrak{so}(7,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$
$E\,VII$		$V = \mathbb{S}^+ \otimes \mathbb{C}^2$ $\mathfrak{der}(V) = \mathfrak{so}(9,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C})$

Exceptional KTS associated to E_7

Special type

$\mathfrak{g}^o$	Satake Diagram	KTS
$E\,V$		$V = \Lambda^3(\mathbb{C}^7)^*$ $\mathfrak{der}(V) = \mathfrak{so}(7,\mathbb{C})$
$E\,VI$		No KTS
$E\,VII$		No KTS

Exceptional KTS of special type $\mathfrak{g} = E_7$

Let η be scalar product on $\mathbb{C}^7$ and $\sharp$ the associated musical isomorphism that sends $\Lambda^3(\mathbb{C}^7)^*$ to $\Lambda^3\mathbb{C}^7$. Let $\bullet$ be the natural projection of $\Lambda^3\mathbb{C}^7 \otimes \Lambda^3(\mathbb{C}^7)^*$ to $\mathfrak{sl}(7, \mathbb{C})$.

Thm VI. The vector space $\Lambda^3(\mathbb{C}^7)^*$ with *triple product*

$$(\alpha\beta\gamma) = \frac{2}{7}\eta(\alpha, \beta)\gamma - (\beta^\sharp \bullet \alpha) \cdot \gamma$$

is a *simple KTS* with associated Lie algebra $\mathfrak{g} = E_7$ and derivation algebra $\mathfrak{so}(7, \mathbb{C})$.

Thanks!