Geometrical Lie Superalgebras and their Applications

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1 Lecture 1

The main theme of this lecture series is the Lie algebras and Lie superalgebras with geometric origins. We will also consider spinor fields satisfying certain PDE Before going to the super-picture, I would like to discuss the classical picture, as it is simpler.

1.1 Lie algebras of isometries of a Riemannian manifolds

Let (M, g) be a Riemannian manifold.

Definition 1. *A vector field ξ is called a Killing vector field if its flow preserves the metric g, or equivalently $\mathcal{L}_\xi g = 0$.*

Let $\zeta \in \Xi(M)$ and denote $A_\zeta : TM \to TM$ given by

$$A_\zeta(x) = -\nabla_X \zeta \tag{1}$$

Lemma 1. *ζ is Killing if A_ζ is an action of $\mathfrak{so}(TM)$.*

Proof.

$$g(A_\zeta Y, Z) = -g(\nabla_Y \zeta, Z) = g([\zeta, Y], Z) - g(\nabla_\zeta Y, Z) = \zeta(g(Y, Z)) - g(Y, [\zeta, Z]) - g(\nabla_\zeta Y, Z) = g(Y, \nabla_\zeta Z) - g($$

□

1.2 Killing as parallel sections

Killing vector fields can be considered as parallel sections of some bundle. Consider

$$E = \mathfrak{so}(TM) \oplus TM$$

and define the covariant derivative

$$D_X(\zeta, A) = (\nabla_X \zeta + A(X), \nabla_X A - R(X, \zeta))$$

Where R is the curvature,

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z \nabla_Y \nabla_X Z$$

Proposition 1 (Kostant 1955)**.** *Parallel sections of E with respect to D are precisely the Killing vector fields.*

Proof. $D_X(\zeta, A) = 0$ if and only if $A(X) = -\nabla_X \zeta$ (i.e. $A = A_\zeta$) and

$$(\nabla_X A)Y = \nabla_X(AY) - A(\nabla_X Y) = -\nabla_X \nabla_Y \zeta + \nabla_{\nabla_X Y}\zeta$$

We take the difference

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X &= -\nabla_X \nabla_Y \zeta + \nabla_{\nabla_X Y}\zeta + \nabla_Y \nabla_X \zeta + \nabla_{\nabla_Y X}\zeta = \\ &= -\nabla_X \nabla_Y \zeta + \nabla_Y \nabla_X \zeta + \nabla_{[X,Y]}\zeta\end{aligned}$$

□

Proposition 2. *Consider the space of parallel sections of E with D. Then the Lie bracket of Killing vector fields is*

$$[(\zeta, A), (\eta, B)] = (A\eta - B\zeta, [A,B] + R(\zeta, \eta))$$

Proof.

$$\begin{aligned}&[(\zeta, A), (\eta, B)] = ([\zeta, \eta], -\nabla[\zeta, \eta]) \\ &\nabla_\zeta \eta - \nabla_\eta \zeta = A\eta - B\zeta\end{aligned}$$

□

Let

$$G = \{\varphi : M \to M \text{ diffeo with } \varphi^* g = g\}$$

and

$$\mathfrak{g} = \{\xi \in \mathcal{D}(M) : \mathcal{L}_\xi g = 0\}$$

Question 1. *What kind of Lie algebra is $\mathfrak{g}$?*

Consider the flat model $M = \mathbb{R}^n$ with g the Euclidean metric. Then

$$\begin{aligned}&G = O(n) \ltimes \mathbb{R}^n \\ &\mathrm{euc}(n) = \mathfrak{so}(n) \ltimes \mathbb{R}^n\end{aligned}$$

$$\begin{aligned}&[A,B] = AB - BA \\ &[A,v] = Av \\ &[v,w] = 0\end{aligned}$$

for $A, B \in \mathfrak{so}(n), v, w \in \mathbb{R}^n$. It is easy to see that this is a graded Lie algebra. There are more flat models. Consider the sphere with the round metric,

$$M = S^n \subset \mathbb{R}^{n+1}$$
$$G = O(n+1)$$
$$\mathfrak{g} = \mathfrak{so}(n+1)$$

Fix $x \in S^n$ and write $V = T_x S^n$, let H be the stabilizer of x in G. Then as a vector space, $g \simeq \mathfrak{so}(V) \oplus V$ and

$$[A, B] = AB - BA$$
$$[A, v] = Av$$
$$[v, w] = \rho(v, w) \in \mathfrak{so}(V)$$

And we have that $\rho = R|_x$ is the curvature. In particular $\mathfrak{g}$ is not a graded Lie algebra, but it is filtered.

Definition 2. *A filtration on a Lie algebra $\mathfrak{g}$ is a sequence of subspaces*

$$\mathfrak{g}$$

missing some here...

Definition 3. *Let $\mathfrak{g}^\bullet$ be a filtration of $\mathfrak{g}$. Then the associated graded Lie algebra $\mathfrak{g}_\bullet$ is*

$$\mathfrak{g}_\bullet = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

where $\mathfrak{g}_n = \mathfrak{g}^n / \mathfrak{g}^{n+1}$

Remark 1. $[\mathfrak{g}^n, \mathfrak{g}^{m+1}] + [\mathfrak{g}^{n+1}, \mathfrak{g}^m \subset \mathfrak{g}^{n+m+1}]]$

Theorem 1. *The Lie algebra $\mathfrak{g}$ of infinitesimal isometries of (M, g) is filtered, and its associated graded Lie algebra is a subalgebra of $\mathfrak{euc}(n)$. We say that $\mathfrak{g}$ is a filtered deformation of $\mathfrak{euc}(n)$.*

2 Lecture 2

Proof. Proof of theorem from previous lecture. We localize Killing vectors at $x \in M$. Set $V = T_x M$.

$$E|_x = V \oplus \mathfrak{so}(V)$$

and $\mathfrak{g}$ is a subspace of $E|_x$. We have a short exact sequence

$$0 \to h \to \mathfrak{g} \to V \to 0$$

and $V' = \{\xi|_x,\ \text{with}\ \xi \in \mathfrak{g}\} \subset V,\ h = \{\xi \in \mathfrak{g}\ \text{with}\ \xi|_x = 0\}$. Hence $\mathfrak{g} \simeq h \oplus V$ as a vector space. Taking the Lie brackets of $\mathfrak{g}$ on $h \oplus V$ gives

$$\begin{aligned}
[A, B] &= AB - BA \\
[A, v] &= Av + \delta(A, v) \in V + h \\
[v, w] &= \alpha(v, w) + \rho(v, w) \in V' + h
\end{aligned}$$

and here

$$\begin{aligned}
\alpha(A, v) &= X_v w - X_w v \\
\delta(A, v) &= [A, X_v] - X_{Av} \\
\rho(v, w) &= [X_v, X_w] - X_{\alpha(v,w)} + R(v, w)
\end{aligned}$$

□

2.1 Rudiments of Spinorial algebra and Spin geometry

Let (V, η) be a vector space with (positive definite) inner product.

Definition 4. *The Clifford algebra $\mathcal{C}l(V)$ associated to (V, η) is the algebra generated by V with the relation $v^2 = -\eta(v, v)1$ for all $v \in \mathcal{C}l(V)$.*

Example 1. *Let $\{e_N\}$ be an orthonormal basis of V. Then $e_i e_j + e_j e_i = -2\delta_{ij}1$.*

We have $\mathcal{C}l(V) \simeq \Lambda V$, but with a different product. $\mathcal{C}l(V)$ has Lie algebra structure via the Clifford commutator and $\mathfrak{so}(V) \simeq \{e_i e_j | i < j\}$, $e_i \wedge e_j \mapsto \frac{1}{2} e_i e_j$. Exponentiating this $\mathfrak{so}(V)$ yields the spin group

$$\mathrm{Spin}(V) = \{g = v_1 \cdots v_2 k \in \mathcal{C}l(V) | v_i \in V, \eta(v_i, v_j) = 1\}$$

If $g \in \mathrm{Spin}(V)$ and $v \in V$ then $gvg^{-1} \in V$. This is the twofold cover Ad : $\mathrm{Spin}(V) \to SO(V)$. Restricting the module S of $\mathcal{C}l(V)$ to $\mathrm{Spin}(V)$ we obtain the spinor module.

2.2 Spin geometry

Let (M^n, g) be an orientable Riemannian manifold and

$$SO(M) = \{u = (e_1, \ldots, e_n)\}$$

be the bundle of oriented orthonormal frames. A spin structure on M is a principal $\mathrm{Spin}(n)$-bundle $\mathrm{Spin}(M) \to M$ together with a bundle morphism $\mathrm{Spin}(M) \to SO(M)$ which restricts fiberwise to Ad. There are topological obstructions to the existence of spin structures. They need not be unique.

Example 2. *Let $M = S^n$ be the homogeneous space $Spin(n+1)/Spin(n)$. If we are in dimension $n \geq 2$ then there exists a unique spin structure. Otherwise there are two nonequivalent choices.*

Definition 5. *The oriented vector bundle $S(M) = Spin(M) \times_{Spin(n)} S$ is the spinor bundle of (M, g).*

Holonomy rep	Geometry	number of parallel spinors
$SU(n)$	CY	2
$Sp(n)$	HK	$n+1$
$G_2 \subset SO(7)$	Exceptional	1
$\mathrm{Spin}(7) \subset SO(8)$	Exceptional	1

Table 1: Complete, simply connected, irreducible Riemannian manifolds with parallel spinors

2.3 Spinor fields satisfying special PDE

Definition 6. *A spinor field $\epsilon \in \Gamma(S(M))$ is*

1. *parallel if $\nabla\epsilon = 0$*
2. *Killing if $\nabla_X \epsilon = \lambda X \cdot \epsilon$, λ is called the Killing constant.*

Theorem 2. *If (M, g) admits a non-trivial parallel spinor, then it is Ricci-flat, $Ric(g) = 0$.*

Proof. We think of the curvature as being $R \in \mathrm{End}(S(M)) \otimes \Lambda^2 T^* M$.

$$\nabla_X \epsilon = 0 \Rightarrow R(X, Y)\epsilon = 0$$

We take the "Clifford trace" of R:

$$0 = 4 \sum_j e_j \cdot R(e_i, e_j)\epsilon$$

Now $R(e_i, e_j) = \frac{1}{2} \sum_{k,l} R_{i,j,k,l} e_k \wedge e_l$ is acting on spinors $R(e_i, e_j) = \frac{1}{4} \sum_{k,l} R_{ijkl} e_k \cdot e_l \cdot \epsilon$. Hence

$$0 = \sum_{j,k,l} R_{ijkl} e_j \cdot e_k \cdot e_l \cdot \epsilon = \sum_{j,k,l} R_{ijkl} (e_{jkl} - \eta_{jk} e_l + \eta_{jl} e_k) \cdot \epsilon = \sum_{j,k,l} R_{ijkl} (e_{jkl} + 2\eta_{jl} e_k) \cdot \epsilon$$

Then we use the algebraic Bianchi identity

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0$$

and we are left with $0 = -\sum_{jkl} R_{ijkl} \eta_j l \cdot \epsilon = -\sum_k \mathrm{Ric}_k e_k \cdot \epsilon$. Thus if we look at the Ricci tensor of TM, we have $\mathrm{Ric}(X) \cdot \epsilon = 0$, thus $g(\mathrm{Ric}(X), \mathrm{Ric}(X))i = 0$ and $\mathrm{Ric}(X) = 0$. □

3 Lecture 3

Before we go to the super world, we will finish the classical story. Yesterday we learned that having a parallel spinor is a sufficient condition to be Ricci-flat.

Theorem 3 (Wang 1989).

Theorem 4. *If (M, g) has a Killing spinor with Killing constant $\lambda \in \mathbb{C}$, then $Ric = 4\lambda^2(n-1)g$, i.e. M is Einstein and λ is real or pure imaginary.*

Let us consider the transpose of Clifford multiplication $V \otimes S \to S$ with respect to η and $\langle , \rangle$. "Dirac aneut" $K : S \otimes S \to V$ given by $\eta(K(s_1, s_2), v) = \langle s_1, v \cdot s_2 \rangle$.

Lemma 2. *Let ϵ_1, ϵ_2 be Killing spinors with the same Killing constant. Then $\xi = K(\epsilon_1, \epsilon_2)$ is a Killing vector.*

Proof.

$$g(\nabla_X \xi, Y) = g(K(\nabla_X \epsilon_1, \epsilon_2), Y) + g(K(\epsilon_1, \nabla_X \epsilon_2), Y) =$$
$$\lambda \langle X \cdot \epsilon_1, Y \cdot \epsilon_2 \rangle + \lambda \langle \epsilon_1, Y \cdot X \cdot \epsilon_2 \rangle =$$
$$\lambda \langle \epsilon_1, (Y \cdot X - X \cdot Y) \cdot \epsilon_2 \rangle$$

□

3.1 Generic contraction of Lie algebras (FOF)

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Definition 7.

$$\mathcal{L}_\xi \epsilon = \nabla_\xi \epsilon + A_\xi \epsilon$$

3.2 Main properties of spinorial Lie derivative

$\xi, \eta \in K_{\bar{o}}$, $X \in \mathcal{D}(M)$, $\epsilon \in \Gamma SM$, $f \in C^\infty(M)$.

1. $[\mathcal{L}_\xi, \mathcal{L}_\eta]\epsilon = \mathcal{L}_{[\xi,\eta]}\epsilon$
2. $\mathcal{L}_\xi(X \cdot \epsilon) = [\xi, X] \cdot \epsilon + X \cdot \mathcal{L}_\xi \epsilon$
3. $\mathcal{L}_\xi(f\epsilon) = \xi(f)\epsilon + f\mathcal{L}_\xi \epsilon$
4. $[\mathcal{L}_\xi, \nabla_X]\epsilon = \nabla_{[\xi,X]}\epsilon$

Lemma 3. *If $\xi \in K_{\bar{o}}$ then $\epsilon \in K_i$ implies $\mathcal{L}_\xi \epsilon \in K_i$*

Proof.

$$\nabla_X \mathcal{L}_\xi \epsilon = \mathcal{L}_\xi \nabla_X \epsilon - \nabla_{[\xi,X]}\epsilon =$$
$$\lambda \mathcal{L}_\xi (X \cdot \epsilon) - \lambda [\xi, X] \cdot \epsilon =$$
$$\lambda X \cdot \mathcal{L}_\xi \epsilon$$

□

Jacobi identities: $\Lambda^3 K \to K$

1. $\Lambda^3 K_{\bar{o}} \to K_{\bar{o}}$ is Jacobi for vector fields
2. $\Lambda^2 K_{\bar{o}} \otimes K_i \to K_i$ holds as $\mathcal{L}$ is a representation
3. $K_{\bar{o}} \otimes \Lambda^2 K_i \to K_{\bar{o}}$ is a computation...
4. $\Lambda^3 K_i \to K_i$ does not hold in general, it is $K_{\bar{o}}$ equivariance. If $(\Lambda^3 K_i^* \otimes K_i)^{K_{\bar{o}}} = 0$ then $K = K_{\bar{o}} \oplus K_i$ is a Lie algebra.

3.3 Normed real division algebras

We have $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These yield corresponding Hopf fibrations:

$$S^1 \to^{S^0} S^1$$
$$S^3 \to^{S^1} S^2$$
$$S^7 \to^{S^3} S^4$$
$$S^{15} \to^{S^7} S^8$$

and $S^\times \subset \mathbb{K}^2$, $S^\times \simeq \mathbb{K}P^1$

Theorem 5. *Applying the Killing algebra construction to the octonionic Hopf fibration, one gets*

$$E_8 = \mathfrak{so}(16) \oplus \mathbb{R}^{128}$$

associated to Killing vectors on S^{15} and Killing spinors on S^{15}, and

$$F_8 = \mathfrak{so}(9) \oplus \mathbb{R}^{16}$$

Corresponding to Killing vectors and spinors on S^7.

4 Lecture 4

The goal is to arrive at describing supergravity today. This will depend on representations of the super-Poincare group. We will see the PDE which describes this situation. I will show an example of how to construct a Killing superalgebra from a supergravity background. We will denote the Poincare algebra by

$$P_{\bar{0}} = \mathfrak{so}(V) \ltimes V$$

The main case of interest today will be $\dim V = 11$. Let $\{e_N\} = \{e_0, e_1, \ldots, e_10\}$ be an orthonormal basis. In $\mathcal{C}(V)$, we have

$$e_0^2 = -1, e_i^2 = 1, i = 1 \ldots 10$$

We have

$$\mathcal{C}(V) \simeq \mathbb{R}(32) \oplus \mathbb{R}(32) \simeq \mathrm{End}(S_+) \oplus \mathrm{End}(S_-)$$

We have two inequivalent representations $S_\pm$, distinguished $vol|_{S_\pm} = \pm 1$. For us, we will let $S = S_-$. On S we have a symplectic form $\langle , \rangle$ satisfying

$$\langle v \cdot e_i, e_j \rangle = -\langle e_i, v \cdot e_j \rangle$$

4.1 Basics about reps of the Poincare group

Relativistic quantum particles in physics are the same as irreducible unitary representations $\mathcal{H}$ of $P(V)$, $P(V) = \mathrm{Spin}(V) \ltimes V$. They are constructed by Wigners "little group" method (a special case of Mackey's theory of induced representations).

$$P^2 = \sum_{\mu,\nu} \eta^{\mu} e_\mu e_\nu \in \mathcal{U}(p_{\bar{0}})^{p_{\bar{0}}}$$

Clearly

$$[e_\mu, P^2] = 0$$
$$[\mathfrak{so}(V), P^2] = 0$$

because η is $\mathfrak{so}(V)$-invariant. Schur's lemma yields that P^2 acts as a scalar on any irreducible representation $\mathcal{H}$. Physically this scalar is the mass m^2 of the particle. ...something about characters on far blackboard... We let the "Little group" be the maximal compact subgroup of the stabilizer,

$$H = \mathrm{Spin}(10) \text{ for } m^2 > 0,$$
$$H = \mathrm{Spin}(9) \text{ for } m^2 = 0,$$

Consider the bundle $\mathrm{Spin}^0(V) \to \mathrm{Spin}^0(V)/H$ over the orbit and finite-dimensional unitary rep U of H. (mdn rep always factors through the little group). Then consider complex Hermitian vector bundle

$$E = \mathrm{Spin}^0(V) \times_H U$$

We will let $\mathcal{H} = L^2(E)$.

Example 3. *We will consider the graviton g, which is massless. This corresponds to*

$$Sym_0^2 W$$

Where W is the euclidean 9-dim rep, and the tensor power is the traceless part. We have a 3-fold potential,

$$A \leftrightarrow \Lambda^3 W$$

and the gravitino

$$\Psi \leftrightarrow (W \otimes \Sigma)_0$$

where Σ *is the spinor rep of* $Spin(9)$ *and the tensor product is once again traceless.*

4.2 An incomplete history of supersymmetry

- 1960's: Is there a larger group than $P(V)$ whose irreps correspond to Poincare reps of different mass and spin?
- 1967: the answer is no.
- 1975: The answer is yes, more or less. This requires the introduction of the Poincare-superalgebra

Definition 8. *A Lie superalgebra is a* $\mathbb{Z}_2$*-graded vector space* $\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *equipped with a bracket satisfying*

1. $[\mathfrak{g}_{\bar{i}}, \mathfrak{g}_{\bar{j}}] \subset \mathfrak{g}_{\overline{i+j}}$
2. $[x, y] = (-1)^{|x||y|}[y, x]$
3. $[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]]$

where $|x| = 0$ *for* $x \in \mathfrak{g}_{\bar{0}}$ *and* $|x| = 1$ *for* $x \in \mathfrak{g}_{\bar{1}}$

The Poincare superalgebra is a graded Lie superalgebra

$$\begin{aligned} &P = \mathfrak{so}(V) \oplus S \oplus V \\ &P_{\bar{0}} = \text{ Poincare algebra} \\ &P_{\bar{1}} = S \end{aligned}$$

$$\begin{aligned} &[A, B] = AB - BA \\ &[A, v] = Av \\ &[v, w] = 0 \\ &[A, s] = As \\ &[s, s] = K(s, s) \\ &[v, s] = 0 \end{aligned}$$

- 1978: There exists an irrep of P for $\dim V = 11$ with field content (g, A, Ψ).
- unreadable

Definition 9. *A supergravity background in 11D is a Lorentzian spin manifold* $(M^1 1, g, F)$ *where* $F \in \Omega^4(M)$ *is a closed form such that the following PDE hold.*

$$\begin{aligned}
&d * F = \frac{1}{2} F \wedge F \text{ Maxwell} \\
&Ric(X,Y) = \frac{1}{2} g(i_X F, i_Y F) - \frac{1}{6} ||F||^2 g(X,Y) \text{ Einstein} \\
&\delta_\epsilon \Psi = D\epsilon + O(\Psi), \epsilon \in \Gamma(S(M)) \\
&D_x \epsilon = \nabla_X \epsilon - \frac{1}{24}(X \cdot F^\# - 3F^\# \cdot X) \cdot \epsilon
\end{aligned}$$

Definition 10. *A Killing spinor is a spinor field* $\epsilon \in \Gamma(S(M))$ *such that* $D\epsilon = 0$.

Example 4. $M = \mathbb{M}$, $F = 0$. *A Killing spinor is parallel with respect to the flat metric and there are 32 independent Killing spinors.*

4.2.1 Classification result of maximal SUSY backgrounds (D - flat)

- $M = \mathbb{M}$, $F = 0$
- $AdS_4 \times S^7$, $F \propto vol(AdS)$
- $AdS_7 \times S^4$, $F \propto vol(S^4)$
- A Lorentzian symmetric space KG of solvable Lie group ($||F||^2 = 0$)

5 Lecture 5

Today we will hear about some more recent results. Full proofs will not be available, as most of them are involved, but we can discuss some ideas. Yesterday we heard about supergravity backgrounds, which are 11D manifolds with a coupled system of PDE. Note that we let objects freely act by clifford multiplication without explicitly using the metric.

Theorem 6 (FOF,Messer, Philip 2005)**.** *: Every supergravity background has a canonically associated Lie superalgebra* $K = K_{\bar{0}} \oplus K_{\bar{1}}$.

Proof. Idea of proof:

$$\begin{aligned}
K_{\bar{0}} &= \{\xi \in \mathcal{D}(M) | \mathcal{L}_\xi g = \mathcal{L}_\xi F = 0\} \\
K_{\bar{1}} &= \{\epsilon \in \Gamma(S(M)) | D\epsilon = 0\}
\end{aligned}$$

Lie brackets are defined by the Dirac current and spinorial Lie derivative. Key points:

1. $K = K(\epsilon, \epsilon)$ is a Killing vector field. Then we use that $i_K F = -d\omega^{(2)}$, where $\omega^{(2)}(X,Y) = \langle \epsilon, (X \wedge Y) \cdot \epsilon \rangle$, and thus $\mathcal{L}_K F = (di_K + i_K d)F = 0$

Background	$K = K_{\bar{0}} \oplus K_{\bar{1}}$
$\mathbb{M}$	P
$AdS_4 \times S^7$	$\mathfrak{osp}(8\|4)$
$AdS_7 \times S^4$	$\mathfrak{osp}(6,2\|4)$
KG	Solvable Lie superalgebra

Table 2: Examples of backgrounds with Lie superalgebras

2. The Jacobi identity with 3 odd elements: $\mathcal{L}_K\epsilon = 0$. $\mathcal{L}_K\epsilon = \nabla_K\epsilon + A_K\epsilon = \frac{1}{24}(K \cdot F - 3F \cdot K) \cdot \epsilon + A_K\epsilon$, $A_K(X) = -\nabla_X(K(\epsilon,\epsilon)) = -2K(\nabla_X\epsilon,\epsilon)$.

□

Theorem 7. *If* $\dim K_{\bar{1}} > 16$*, then* (M, g, F) *is locally homogeneous.*

Proof. (M, g, F) is locally homogeneous if and only if the evalutation $K_{\bar{0}} \to^{ev_x} T_xM$ is surjective for all $x \in M$.

Claim 1. $[K_{\bar{1}}, K_{\bar{1}}] \to^{ev_x} T_x$ *is surjective.*

In other words if $S' \subset S$ with $\dim S' > 16$ then $[S', S'] = V$. If this is not true then there exists nonzero $v \in V$ such that $v \perp [S', S']$, i.e.

$$0 = \eta(v, [s_1, s_2]) = \langle s_1, v \cdot s_2\rangle \Rightarrow v|_{S'} : S' \to (S')^{\perp}$$

Now $\dim(S')^{\perp} < 16$, hence $v\cdot$ has a kernel and v is lightlike. Thus $[S', S']$ is a vector space consisting of lightlike vectors and $\dim([S', S'])^{\perp} = 1$ and generated by v. Fact: Dirac current $[s, s]$ is causal (lightlike or timelike). Thus $[s_1, s_2] \subset \mathbb{R}v \Rightarrow [S', S'] \subset \mathbb{R}v$, a contradiction. □

Theorem 8 (FOF, S)**.** *:* $K = K_{\bar{0}} \oplus K_{\bar{1}}$ *is a filtered subdeformation of the Poincare supealgebra* P*.*

Proof. "Killing supertransport".

$$\begin{aligned} F &= F_{\bar{0}} \oplus F_{\bar{1}} \\ F_{\bar{0}} &= TM \oplus \mathfrak{so}(TM) \\ F_{\bar{1}} &= S(M) \\ D_X(\xi, A, \epsilon) &= (\nabla_X\xi + A(X), \nabla_X A - R(X,\xi), D_X\epsilon) \end{aligned}$$

So that

$$K \simeq \{D \text{ parallel spinors on } E\}$$

We then localize at $x \in M$ and track back Lie brackets on $h \oplus S' \oplus V'$, where

$$\begin{aligned} h &= \{\xi \in K_{\bar{0}} | \xi|_x = 0\} \\ S' &= \{\epsilon|_x \text{ where } \epsilon \in K_{\bar{1}}\} \\ V' &= \{\xi|_x \text{ where } \xi \in K_{\bar{0}}\} \subset S \end{aligned}$$

We get the bracket deformations

$$
\begin{aligned}
&[A,B] = AB - BA \\
&[A,s] = As \\
&[v,s] = \beta(v,s) \in S' \\
&[A,v] = Av + \delta(A,v) \\
&[s,s] = K(s,s) + \gamma(s,s) \\
&[v,w] = \alpha(v,w) + \rho(v,w)
\end{aligned}
$$

Where δ, γ, ρ go to h and β goes to S', α goes to V', and α, δ, ρ are as in classical case.

$$
\begin{aligned}
&\beta(v,s) = \beta^{\varphi}(v,s) + X_v(s) \\
&\gamma(s,s) = \gamma^{\varphi}(s,s) - X_{K(s,s)} \\
&\beta^{\varphi}(v,s) = \frac{1}{24}(v \cdot \varphi - 3\varphi \cdot v) \cdot s \\
&\gamma^{\varphi}(s,s)(v) = -2K(\beta^{\varphi}(v,s), s) \\
&\varphi = F|_x \in \Lambda^4 V^* \\
&X : V' \to \mathfrak{so}(V)
\end{aligned}
$$

Section of the splitting of Killing vector fields □

Theorem 9. *(FOF,S): Let* (M^{11}, g, F) *be a a Lorentzian spin manifold with closed* $F \in \Omega^4(M)$*. If* $\dim K_{\bar{1}} > 16$ *then the Einstein and Maxwell equations are unconditionally satisfied.*

Proof. Sketch of proof: Let $\epsilon \in \Gamma(S(M))$ and define forms:

$$
\begin{aligned}
&\omega^{(1)}(X) = \langle \epsilon, X \cdot \epsilon \rangle \\
&\omega^{(2)}(X,Y) = \langle \epsilon, (X \wedge Y) \cdot \epsilon \rangle \\
&\omega^{(5)}(X_1, \ldots, X_5) = \langle \epsilon, (X_1 \wedge \cdots \wedge X_5) \cdot \epsilon \rangle
\end{aligned}
$$

Nontrivial facts: If $D\epsilon = 0$, then $K = K(\epsilon, \epsilon)$

- $d\omega^{(2)} = -i_K F$
- $d\omega^{(5)} = i_K * F - \omega^{(2)} \wedge F$

Let us compute.

$$
\begin{aligned}
0 = *\mathcal{L}_K F = \mathcal{L}_K * F = di|_K * F + i_K d * F = d(\omega^{(2)} \wedge F) + i_K d * F = \\
= -\frac{1}{2} i_K (F \wedge F) + i_K d * F = i_K (d * F - \frac{1}{2} F \wedge F)
\end{aligned}
$$

Thus if $\dim K_{\bar{1}} > 16$ then the Maxwell equation holds. Remember now that the Jacobi identity is given.

$$\mathrm{Ric}(v, K(s,s)) = \frac{1}{2} F^2_{ab} v^a (e^b \cdot s, s) + \frac{1}{6} ||F||^2 \langle v \cdot s, s \rangle + \frac{1}{6} \langle (v \wedge F \wedge F + 2 i_v dF - v \wedge dF) \cdot s, s \rangle$$

Einstein equation is unconditionally satisfied. □