MAT-3110: Differential Geometry – 1

Theoretical questions:

- Manifolds: Definition (charts, atlas). Examples (S^n, $\mathbf{R}P^n$, $\mathbf{C}P^n$).
- Smooth maps $F:M\to N$ and diffeomorphisms. Local coordinates.
- Category of smooth manifolds. Morphisms $C^\infty(M,N)$.
- Algebra $A=C^\infty(M)$. Induced morphism $F^*: C^\infty(N)\to C^\infty(M)$. Algebraization functor (contravariant) to the category of commutative algebras. Que: $\mathbf{1}^*$?
- Points and ideals. Maximal ideal $\mathfrak{m}_x$ (exact 3-sequence). $Spec(A)$ and Gelfand thm.
- Space of jets $J^k{}_x(M)$. Structure of commutative algebra.
  Induced map $[F]_x{}^k: J^k{}_y(N)\to J^k{}_x(M)$, $y=F(x)$.
- Jet-spaces, projections $\pi_{k,k-1}: J^kM\to J^{k-1}M$ and the corresponding exact 3-sequence.
- Hadamard's lemma. Equivalence of 2 descriptions of $\mathfrak{m}_x{}^k$ – algebraic and via Taylor-Maclaurin series.
- Cotangent $T_x{}^*M$ and tangent T_xM spaces via jets. Differential $d_xF:T_xM\to T_yN$. (to defs – dual to $d_x{}^*F$ and via velocities of curves). Relation to the Jacobian $J(F)$.
- Lie groups and algebras. Examples: $GL(n,\mathbf{R})$, $SL(n,\mathbf{R})$, $SO(n)$, $GL(n,\mathbf{C})$, $U(n)$.
- Lie algebras – general definition. Tangent space $\mathfrak{g}=T_eG$. Exponential map. Calculation of Lie algebras of the above Lie groups.
- Vector fields $\mathfrak{D}(M)$. Structure of A-module and of (infinite-dim) Lie algebra.
- Trajectories of vector fields (flow). Relation to ODEs (loc.coordinates).
- Lie derivative L_X – action on functions, vector fields, differential forms. Properties of L_X.
- Differential 1-forms. Expression in local coordinates. Pairings with vector fields.
- Differential k-forms and de Rham differential d (for 1-forms relation to $Hess(f)$). Properties of d and de Rham complex.
- Tensors: algebraic theory. Categorical approach and universal property. Symmetric and skew-symmetric tensor products.
- Spaces T^kV, T^kV^*, S^kV, Λ^kV and their dimensions. Tensor algebras.
- Decomposition of a 2-tensor into symmetric and skew-symmetric parts.
- Decomposable k-forms. Cartan criterion for decomposability of a 2-form.
- Map $\Lambda^nF: \Lambda^nM\to\Lambda^nM$. Relation to the determinant.
- Evaluations in the tensor algebra. Inner derivation i_X.
- Derivations $\Delta:\Omega^k(M)\to\Omega^{k+1}(M)$. Commutator of graded derivations.
- Infinitesimal Stokes formula. Application to symmetries X_f of $J^1(\mathbf{R}^n)$ with Cartan form $\omega=du-p_idx^i$ (contact structure).
- Cartan's formula for differential $d\omega$ of 1- and 2-forms.

Practice:

- Calculation of $d\omega$, $L_X\omega$, $i_X\omega$, $[\xi,\eta]$ in concrete examples.
- Calculation of tensor, symmetric and wedge produces.
- Understanding symmetric and skew-symmetric products of linear operators.
- Ability to realize if a differential 2-form is decomposable, 1-form exact etc.