Differential Geometry and Mathematical Physics II – 2011

Syllabus

1. Vector Bundles and Geometric Structures:

- Different definitions (including cocycle def). Examples: trivial VB, Möbius bundle, tautological bundles $\xi_n^1$ over $\mathbb{R}P^n$ and $\mathbb{C}P^n$.
- Morphisms and isomorphisms of VB. Examples of isomorphic and non-isomorphic VB. Module of Sections. Criterion of triviality.
- Constructions of VB: induced VB, Whitney sum, tensor product, symmetric and skew-symmetric products, dual and quotient bundle.
- VB with structure group $G$. Oriented VB, Riemannian VB. Two different definitions. Examples of non-orientable VB.
- Constructions of Riemannian VB. VB over a paracompact manifold is Riemannian. Realization of the quotient as a subbundle in Riemannian VB.
- Differential forms, vector fields and Riemannian metrics as sections.
- Other geometric structures: almost complex, almost symplectic, almost Hermitian.
- Distributions. Definition by means of vector fields and covector fields. Curvature of the distribution: two definitions. Frobenius Theorem.

2. Covariant differentiation:

- Linear connection. Three definitions: derivation $\nabla$ (Koszul connection), Parallel transport (equation of the transport) and Horizontal distribution (Ehresmann connection).
- Equivalence of these definitions (including formulation and proof of the *important* lemma on morphisms of sections).
- Constructions of new connections: induced connection, dual connection, Whitney sum, tensor product etc.
- VB over paracompact manifold possesses a linear connection. Existence of connections preserving geometric structures.
- Trivial connections and gauge transformations.

3. Curvature of geometric structures:

- Curvature of a linear connection. Criterion of the connection to be locally trivial.
- Holonomy group, holonomy algebra and its relation to the curvature.
- Monodromy of locally flat connections. Criterion of the connection to be globally trivial.

- Connection on manifolds. Torsion. Existence of symmetric connection.
- Existence of symmetric connection preserving Riemannian metric (Levi-Civita connection) with proof.
- Equation of geodesics. Definition of Riemannian curvature and its properties. Flatness.
- Curvature for almost complex and almost symplectic structures.

4. Symplectic and Contact Geometry:

- Linear symplectic geometry. Rank of 2-form and Cartan normal form.
- Skew-orthogonality. Isotropic, co-isotropic, symplectic, Lagrangian subspaces.
- Linear Darboux theorem (with proof). Linear Weinstein theorem (without proof).
- Group $\mathrm{Sp}(V)$, algebra $\mathrm{sp}(V)$. Isomorphism $\mathrm{sp}(V) \simeq \Lambda^2 V^*$. Lagrangian Grassmanian.
- Symplectic manifolds. Examples. Canonical symplectic structure on $T^*M$.
- Lagrange submanifolds. Weinstein theorem (no proof).
- Symplectomorphisms. Symplectic vector fields = Hamiltonian fields $X_H$.
- Poisson bracket. Iso of Lie algebras $(\mathrm{symp}(M,\Omega); [,])$ and $(C^\infty(M); \{,\})$.
- Non-strict and strict contact structures. Examples of contact manifolds.
- Darboux theorems for symplectic and (strict) contact structures (no proofs).
- Contactomorphisms (examples), contact vector fields. Vector fields $X_F, Y_F$.
- Different brackets in contact geometry. PDEs of 1st order and the methods of Cauchy characteristics.

Technical tools.

**Be able to:** Distinguish between embedding and immersion; Calculate with differential forms and vector fields; Calculate geodesics on the plane, and geodesics on the sphere; Calculate Poisson bracket, check if a function is an integral of the Hamiltonian system; Calculate curvature of a distribution and its graded nilpotent Carnot-Tanaka algebra; Write compatibility for overdetermined PDE systems in Frobenius form, solve PDE by the method of characteristics.