On C-class equations

Dennis The*

Department of Mathematics & Statistics
University of Tromsø

(joint work with Andreas Čap & Boris Doubrov)

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C-class ODE

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> GOAL: Identify C-classes for higher-order ODE (up to $\mathfrak{C}$).

Why study C-class ODE?

- Various classes of ODE $\mathcal{E}$ (up to $\mathfrak{G}$) admit an equiv. descrip. via a canonical Cartan geometry $(\mathcal{G} \to \mathcal{E}, \omega)$ of type (G, P).

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N.B. Existence of Cartan connections is not guaranteed for arb. $\mathfrak{G}$.

Some results

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Rmk: Wilczynski-flat ODE inherit a natural geometric structure on their soln space, e.g. conformal structure when $(m, n) = (1, 2)$.

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$\therefore$ Given ODE is of C-class.

Some non-flat C-class systems examples

Let $\mathbf{u}$ have m components. The following are C-class examples:

The equation for circles in Euclidean space

$$\mathbf{u}''' = 3\mathbf{u}'' \frac{\langle \mathbf{u}', \mathbf{u}'' \rangle}{1 + \langle \mathbf{u}', \mathbf{u}' \rangle}$$

is Wilczynski-flat (Medvedev 2011).

$$\mathbf{u}^{(n+1)} = \mathbf{f}, \quad \text{where} \quad f_i = \begin{cases} 0, & i \neq m; \\ (u_1^{(n)})^2, & i = m \end{cases}$$

has trivializable linearization.

From ODE to filtered manifolds

$J^{n+1}(\mathbb{R},\mathbb{R})$: $(t, u_0, u_1, u_2, ..., u_{n+1})$, contact system:

$$\langle du_0 - u_1 dt, \quad ..., \quad du_n - u_{n+1} dt \rangle.$$

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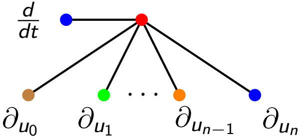
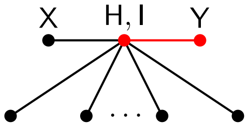
Splitting $\rightsquigarrow G_0 \subset Aut_{gr}(\mathfrak{m})$. Get "filtered G_0-structure of type $\mathfrak{m}$":



Homogeneous models

The contact sym alg of $u^{(n+1)} = 0$ (order ≥ 4) leads to $G = GL_2 \ltimes \mathbb{V}_n$, where $\mathbb{V}_n \cong S^n\mathbb{R}^2$, and $P = LT_2$ in red below.



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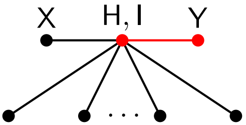
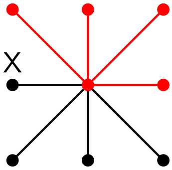


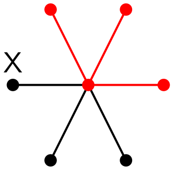
Special (parabolic) cases:

order = 3



$\mathfrak{g} = \mathfrak{sp}_4$

order = 2 (point transf.)



$\mathfrak{g} = \mathfrak{sl}_3$

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A Cartan geometry $(\mathcal{G} \to M, \omega)$ of type (G, P) consists of:

- (right) principal P-bundle $\mathcal{G} \to M$
- Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$, i.e.
  - $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, $\forall u \in \mathcal{G}$;
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- Cartan connection $\omega : T\mathcal{G} \to \mathfrak{g}$, i.e.
  - $\omega_u : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism, $\forall u \in \mathcal{G}$;
  - $(r^p)^*\omega = \mathrm{Ad}_{p^{-1}} \circ \omega$, $\forall p \in P$;
  - $\omega(\widetilde{A}) = A$, $\forall A \in \mathfrak{p}$, where $\widetilde{A}_u = \frac{d}{d\epsilon}\Big|_{\epsilon=0} u \exp(\epsilon A)$.

Curvature: $K = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathcal{G}; \mathfrak{g})$. This is horizontal and completely obstructs flatness, i.e. local equiv to $(G \to G/P, \omega_G)$.

Curv. fcn: $\kappa : \mathcal{G} \to \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, $\kappa(x, y) = K(\omega^{-1}(x), \omega^{-1}(y))$.

- $TM = \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. P-inv. data on $\mathfrak{g}/\mathfrak{p} \rightsquigarrow$ geo. str. on TM.
- Need normalization conditions on K to get an equivalence of categories with underlying structures on M,
  e.g. Riem. geom. $\leftrightarrow$ Cartan geom. of type $(\mathbb{E}(n), O(n))$ with $\mathrm{im}(\kappa) \subset \bigwedge^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}$ (torsion-free).

Descending to the solution space

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Parabolic analogy: Correspondence and twistor spaces (Čap, 2005). Here, it is sufficient to test harmonic curvature κ_H.

Example 1: scalar 3rd order ODE

For $y''' = f(x, y, y', y'')$, have the relative $\mathfrak{C}$-invariants:

$$I_1 = \text{Wünschmann invariant}, \quad I_2 = f_{y''y''y''y''}$$

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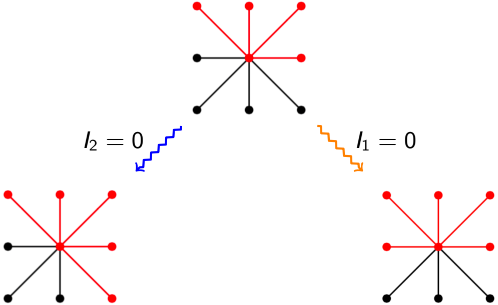
These comprise κ_H. Geometric interpretation:

- $I_2 = 0$: Get a 3-dim contact projective structure on $\mathcal{E}/F$;
- $I_1 = 0$: Get a 3-dim conf. str. on $\mathcal{S} \cong \mathcal{E}/E$ (C-class).



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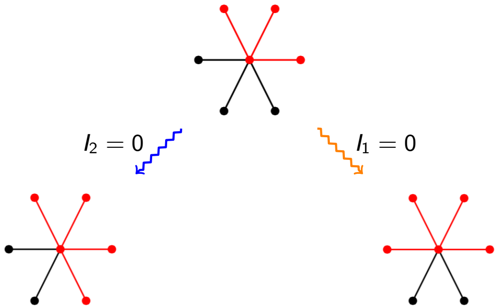
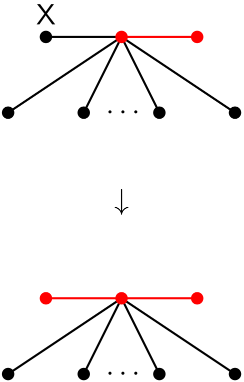
- $I_2 = 0$: geodesic eqn for a 2-dim projective connection.
- $I_1 = 0$: dual 2nd order ODE is a geodesic eqn (C-class).



Model fibration for higher-order ODE



ODE $\mathcal{E}$ up to $\mathfrak{C}$

Solution space $\mathcal{S}$ is equipped with a GL_2-structure (ODE systems: Segré structure modelled on $\mathrm{Seg}(\nu_n(\mathbb{P}^1) \times \mathbb{P}^{m-1}) \hookrightarrow \mathbb{P}(\mathbb{V}_n \otimes \mathbb{R}^m))$

Facts from parabolic geometry theory

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Does $i_X \kappa_E = 0$ imply $i_X \kappa = 0$?

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- Focus on hom. ℓ-component to correct K_1 and K_2. Get new $K_2 = \partial^* \psi$ with ψ of hom. $\geq \ell + 1$. Iterate until $K_2 = 0$.

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These models are homogeneous and have contact sym alg $A_2 \cong \mathfrak{sl}_3$ and $C_2 \cong \mathfrak{sp}_4$.

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Classical fact: The submax sym dim for scalar ODE is usually 2 less than the max, except for 5th and 7th order (where it is 1 less). For the exceptions, the unique submax sym models are:

$$9(u'')^2 u^{(5)} - 45 u'' u''' u'''' + 40 (u''')^3 = 0;$$

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Curvature $\kappa(x, y) = \iota([x, y]) - [\iota(x), \iota(y)]$ is normal and $i_X \kappa = 0$.

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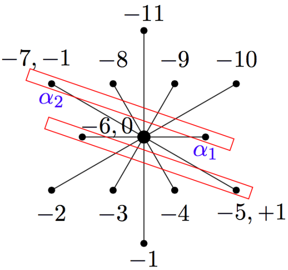

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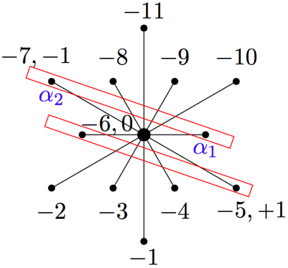

Since $[\mathfrak{s}_{\alpha_1+\alpha_2}, \mathfrak{s}_{2\alpha_1+\alpha_2}] = \mathfrak{s}_{3\alpha_1+2\alpha_2}$, i.e. $(-8, -9) \to -11$, this does not come from an (11th order) ODE.