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Betti numbers of skeletons

Jan Roksvold,* Hugues Verdure

UiT The Arctic University of Norway, Department of Mathematics and Statistics,
N-9037 Tromsø, Norway

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Abstract

We demonstrate that the Betti numbers associated to an $\mathbb{N}_0$-graded minimal free resolution of the Stanley-Reisner ring $S/I_{\Delta^{(d-1)}}$ of the $(d-1)$-skeleton of a simplicial complex Δ of dimension d can be expressed as a $\mathbb{Z}$-linear combination of the corresponding Betti numbers of Δ. An immediate implication of our main result is that the projective dimension of $S/I_{\Delta^{(d-1)}}$ is at most one greater than the projective dimension of S/I_Δ, and it thus provides a new and direct proof of this. Our result extends immediately to matroids and their truncations. A similar result for matroid elongations can not be hoped for, but we do obtain a weaker result for these.

1 Introduction

In this paper we investigate certain aspects of the relationship between an $\mathbb{N}_0$-graded minimal free resolution of the Stanley-Reisner ring of a simplicial complex and those associated to its skeletons. Our main result is Theorem 3.1, which says that each of the Betti numbers associated to an $\mathbb{N}_0$-graded minimal free resolution of $S/I_{\Delta^{(d-1)}}$, where $I_{\Delta^{(d-1)}}$ is the ideal generated by monomials corresponding to

*Corresponding author. E-mail address: `jan.n.roksvold@uit.no`

nonfaces of the $(d-1)$-skeleton of a finite simplicial complex Δ, can be expressed as a $\mathbb{Z}$-linear sum of the Betti numbers associated to S/I_Δ.

Previous results on the Stanley-Reisner rings of skeletons include the classic [8, Corollary 2.6] which states that

$$\operatorname{depth} S/I_\Delta = \max\{j : \Delta^{(j-1)} \text{ is Cohen-Macauley}\}. \tag{1}$$

This result was later generalized to monomial ideals in [6, Corollary 2.5]. By the Auslander-Buchsbaum identity, it follows from (1) that

$$\operatorname{p.d.} I_\Delta \leq \operatorname{p.d.} S/I_{\Delta^{(d-1)}} \leq 1 + \operatorname{p.d.} S/I_\Delta.$$

From the latter of these inequalities it is easily demonstrated, again by using the Auslander-Buchsbaum identity, that every skeleton of a Cohen-Macauley simplicial complex is Cohen-Macauley - a fact which was proved in [8, Corollary 2.5] as well.

That $\operatorname{p.d.} S/I_{\Delta^{(d-1)}} \leq 1 + \operatorname{p.d.} S/I_\Delta$ can also be seen as an immediate consequence of our main result, and Theorem 3.2 thus provides a new and direct proof of this and therefore also of the fact that the Cohen-Macauley property is inherited by skeletons.

The projective dimension of Stanley-Reisner rings has seen recent research interest. Most notably, it was demonstrated in [12, Corollary 3.33] that

$$\operatorname{p.d.} S/I_\Delta \geq \max\{|C| : C \text{ is a circuit of the Alexander dual } \Delta^* \text{ of } \Delta\},$$

with equality if S/I_Δ is sequentially Cohen-Macauley.

Our main result extends immediately to a matroid M and its truncations. Such matroid truncations have themselves seen recent research interest. An example of this being [10], which contains the strengthening of a result by Brylawski [4, Proposition 7.4.10] concerning the representability of truncations.

Corresponding to our main result applied to matroid truncations, we give a considerably weaker result concerning matroid elongations. It says that the Betti table associated to the elongation of M to rank $r(M)+1$ is equal to the Betti table obtained by removing the second column from the Betti table of S/I_M - but only in terms of zeros and nonzeros.

1.1 Structure of this paper

- In Section 2 we provide definitions and results used later on.

- In Section 3 we demonstrate that the Betti numbers associated to a $\mathbb{N}_0$-graded minimal free resolution of the Stanley Reisner ring of a skeleton can be expressed as a $\mathbb{Z}$-linear combination of the corresponding Betti numbers of the original complex. This leads immediately to a new and direct proof that the property of being Cohen-Macauley is inherited from the original complex.

- In Section 4 we see how our main result applies to truncations of matroids. We also explore whether a similar result can be obtained for matroid elongations.

2 Preliminaries

2.1 Simplicial complexes

Definition 2.1. A *simplicial complex* Δ on $E = \{1, \ldots, n\}$ is a collection of subsets of E that is closed under inclusion.

We refer to the elements of Δ as the *faces* of Δ. A *facet* of Δ is a face that is not properly contained in another face, while a *nonface* is a subset of E that is not a face.

Definition 2.2. If $X \subseteq E$, then $\Delta_{|X} = \{\sigma \subseteq X : \sigma \in \Delta\}$ is itself a simplicial complex. We refer to $\Delta_{|X}$ as the *restriction of* Δ *to* X.

Definition 2.3. Let m be the cardinality of the largest face contained in $X \subseteq E$. The *dimension* of X is $\dim(X) = m - 1$.

In particular, the dimension of a face σ is equal to $|\sigma| - 1$. We define $\dim(\Delta) = \dim(E)$, and refer to this as the dimension of Δ.

Definition 2.4 (The i-skeleton of Δ)**.** For $0 \leq i \leq \dim(\Delta)$, let the i-skeleton $\Delta^{(i)}$ be the simplicial complex

$$\Delta^{(i)} = \{\sigma \in \Delta : \dim(\sigma) \leq i\}.$$

In particular, we have $\Delta^{(d)} = \Delta$. The 1-skeleton $\Delta^{(1)}$ is often referred to as the underlying graph of Δ.

Remark. Whenever $\sigma \in \mathbb{N}_0^n$ the expression $|\sigma|$ shall signify the sum of the coordinates of σ. When, on the other hand, $\sigma \subseteq \{1 \ldots n\}$, the expression $|\sigma|$ denotes the cardinality of σ.

2.2 Matroids

There are numerous equivalent ways of defining a matroid. It is most convenient here to give the definition in terms of independent sets. For an introduction to matroid theory in general, we recommend e.g. [13].

Definition 2.5. A *matroid* M consists of a finite set E and a non-empty set $I(M)$ of subsets of E such that:

- $I(M)$ is a simplicial complex.
- If $I_1, I_2 \in I(M)$ and $|I_1| > |I_2|$, then there is an $x \in I_1 \setminus I_2$ such that $I_2 \cup x \in I(M)$.

The elements of $I(M)$ are referred to as the *independent sets* (of M). The *bases* of M are the independent sets that are not contained in any other independent set; in other words, the facets of $I(M)$. Conversely, given the bases of a matroid, we find the independent sets to be those sets that are contained in a basis. We denote the bases of M by $B(M)$. It is a fundamental result that all bases of a matroid have the same cardinality, which implies that $I(M)$ is a *pure* simplicial complex.

The dual matroid $\overline{M}$ is the matroid on E whose bases are the complements of the bases of M. Thus

$$B(\overline{M}) = \{E \setminus B : B \in B(M)\}.$$

Definition 2.6. For $X \subseteq E$, the rank function r_M of M is defined by

$$r_M(X) = \max\{|I| : I \in I(M), I \subseteq X\}.$$

Whenever the matroid M is clear from the context, we omit the subscript and write simply $r(X)$. The rank $r(M)$ of M itself is defined as $r(M) = r_M(E)$. Whenever $I(M)$ is considered as a simplicial complex we thus have $r(X) = \dim(X) + 1$ for all $X \subseteq E$, and $r(M) = \dim(I(M)) + 1$.

Definition 2.7. If $X \subseteq E$, then $\{I \subseteq X : I \in I(M)\}$ form the set of independent sets of a matroid $M_{|X}$ on X. We refer to $M_{|X}$ as the *restriction of M to X*.

Definition 2.8 (Truncation)**.** The i^{th} truncation $M^{(i)}$ of M is the matroid on E whose independent sets consist of the independent sets of M that have rank less than or equal to $r(M) - i$. In other words

$$I(M^{(i)}) = \{X \subseteq E : r(X) = |X|, r(X) \leq r(M) - i\}.$$

Observe that $M^{(i)} = I(M)^{(r(M)-i-1)}$, whenever $I(M)$ is considered as a simplicial complex. That is, the i^{th} truncation corresponds to the $(d-i)$-skeleton.

Definition 2.9 (Elongation)**.** For $0 \leq i \leq n - r(M)$, let $M_{(i)}$ be the matroid whose independent sets are $I(M_{(i)}) = \{\sigma \in E : n(\sigma) \leq i\}$.

Since $r(M_{(i)}) = r(M) + i$, the matroid $M_{(i)}$ is commonly referred to as the *elongation* of M to rank $r(M)+i$. It is straightforward to verify that for $i \in [0,\ldots,n-r(M)]$ we have $\overline{M}_{(i)} = \overline{M^{(i)}}$.

2.3 The Stanley-Reisner ideal, Betti numbers, and the reduced chain complex

Let Δ be an abstract simplicial complex on $E = \{1,\ldots,n\}$. Let $\Bbbk$ be a field, and let $S = \Bbbk[x_1,\ldots,x_n]$. By employing the standard abbreviated notation

$$x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n} = \mathbf{x}^{\mathbf{a}}$$

for monomials, we establish a $1-1$ connection between monomials of S and vectors in $\mathbb{N}_0^n$. Furthermore, identifying a subset of E with its indicator vector in $\mathbb{N}_0^n$ (as is done in Definition 2.10 below) thus provides a $1-1$ connection between squarefree monomials of S and subsets of E.

Definition 2.10. Let I_Δ be the ideal in S generated by monomials corresponding to nonfaces of Δ. That is, let

$$I_\Delta = \langle \mathbf{x}^\sigma : \sigma \notin \Delta \rangle.$$

We refer to I_Δ and S/I_Δ, respectively, as the *Stanley-Reisner ideal* and *Stanley-Reisner ring* of Δ.

Being a (squarefree) monomial ideal, the Stanley-Reisner ideal, and thus also the Stanley-Reisner ring, permits both the standard $\mathbb{N}_0$-grading and the standard $\mathbb{N}_0^n$-grading. For $\mathbf{b} \in \mathbb{N}_0^n$ let $S_\mathbf{b}$ be the 1-dimensional $\Bbbk$-vector space generated by $\mathbf{x}^\mathbf{b}$, and let $S(\mathbf{a})$, S shifted by $\mathbf{a}$, be defined by $S(a)_\mathbf{b} = S_{\mathbf{a}+\mathbf{b}}$. Analogously, for $j \in \mathbb{N}_0$ let S_i be the $\Bbbk$-vector space generated by monomials of degree i, and let $S(j)$ be defined by $S(j)_i = S_{i+j}$. For the remainder of this section let N be an $\mathbb{N}_0^n$-graded S-module.

Definition 2.11. An *($\mathbb{N}_0^n$- or $\mathbb{N}_0$-)graded minimal free resolution* of N is a left complex

$$0 \longleftarrow F_0 \xleftarrow{\phi_1} F_1 \xleftarrow{\phi_2} F_2 \longleftarrow \cdots \xleftarrow{\phi_l} F_l \longleftarrow 0$$

with the following properties:

- $F_i = \begin{cases} \bigoplus_{\mathbf{a}\in\mathbb{N}_0^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}, \mathbb{N}_0^n\text{-graded resolution} \\ \bigoplus_{j\in\mathbb{N}_0} S(-j)^{\beta_{i,j}}, \mathbb{N}_0\text{-graded resolution} \end{cases}$
- $\operatorname{im}\phi_i = \ker\phi_{i-1}$ for all $i \geq 2$, and $F_0/\operatorname{im}\phi_1 \cong N$ (Exact)
- $\operatorname{im}\phi_i \subseteq \mathbf{m}F_{i-1}$ (Minimal)
- $$\phi_i\big((F_i)_{\mathbf{a}}\big) \subseteq (F_{i-1})_{\mathbf{a}} \text{ (Degree preserving, } \mathbb{N}_0^n\text{-graded case)}$$
  $$\phi_i\big((F_i)_j\big) \subseteq (F_{i-1})_j \text{ (Degree preserving, } \mathbb{N}_0\text{-graded case).}$$

It follow from [7, Theorem A.2.2] that *the Betti numbers associated to a ($\mathbb{N}_0$- or $\mathbb{N}_0^n$-graded) minimal free resolution are unique*, in that any other minimal free resolution must have the same Betti numbers. We may therefore without ambiguity refer to $\{\beta_{i,\mathbf{a}}(N;\Bbbk)\}$ and $\{\beta_{i,j}(N;\Bbbk)\}$, respectively, as the $\mathbb{N}_0^n$-graded and $\mathbb{N}_0$-graded Betti numbers of N (over $\Bbbk$). Observe that

$$\beta_{i,j}(N;\Bbbk) = \sum_{|\mathbf{a}|=j} \beta_{i,\mathbf{a}}(N;\Bbbk)$$

where $|\mathbf{a}| = a_1 + a_2 + \cdots + a_n$ (see Remark 2.1, above). Note also that for an $\mathbb{N}_0^n$-graded (that is, monomial) ideal $I \subseteq S$, we have $\beta_{i,\sigma}(S/I;\Bbbk) = \beta_{i-1,\sigma}(I;\Bbbk)$ for all $i \geq 1$, and $\beta_{0,\sigma}(S/I;\Bbbk) = \begin{cases} 1, \sigma = \emptyset \\ 0, \sigma \neq \emptyset \end{cases}$.

The $\mathbb{N}_0$-graded Betti numbers of N may be compactly presented in a so-called *Betti table*:

$$\beta[N](\Bbbk) = \begin{array}{r|cccc} & 0 & 1 & \cdots & l \\ \hline j & \beta_{0,j}(N;\Bbbk) & \beta_{1,j+1}(N;\Bbbk) & \cdots & \beta_{l,j+l}(N;\Bbbk) \\ j+1 & \beta_{0,j+1}(N;\Bbbk) & \beta_{1,j+2}(N;\Bbbk) & \cdots & \beta_{l,j+l+1}(N;\Bbbk) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ k & \beta_{0,k}(N;\Bbbk) & \beta_{1,k+1}(N;\Bbbk) & \cdots & \beta_{l,k+l}(N;\Bbbk) \end{array}$$

By the (graded) *Hilbert Syzygy Theorem* we have $F_i = 0$ for all $i \geq n$. If $F_l \neq 0$ but $F_i = 0$ for all $i > l$, we refer to l as the *length* of the minimal free resolution. It can be seen from e.g. [5, Corollary 1.8] that the length of a minimal free resolution of N equals its projective dimension (p. d. N).

A sequence $f_1, \ldots, f_r \in \langle x_1, x_2, \ldots, x_n \rangle$ is said to be a *regular N-sequence* if f_{i+1} is not a zero-divisor on $N/(f_1 N + \cdots + f_i N)$.

Definition 2.12. The *depth* of N is the common length of a longest regular N-sequence. Whenever N is $\mathbb{N}_0$-graded the polynomials may be assumed to be homogeneous.

In general we have depth $N \leq \dim N$, where $\dim N$ denotes the Krull dimension of N. The following is a particular case of the famous *Auslander-Buchsbaum Theorem*.

Theorem 2.1 (Auslander-Buchsbaum)**.**

$$\text{p.\,d.}\, N + \operatorname{depth} N = n.$$

Proof. See e.g. [7, Corollary A.4.3]. □

Note that the Krull dimension $\dim S/I_\Delta$ of S/I_Δ is one more than the dimension of Δ (see [7, Corollary 6.2.2]). The simplicial complex Δ is said to be *Cohen-Macauley* if depth $S/I_\Delta = \dim S/I_\Delta$. That is, if S/I_Δ is Cohen-Macauley as an S-module.

Definition 2.13. Let $\mathscr{F}_i(\Delta)$ denote the set of i-dimensional faces of Δ. That is,

$$\mathscr{F}_i(\Delta) = \{\sigma \in \Delta : |\sigma| = i+1\}.$$

Let $\Bbbk^{\mathscr{F}_i(\Delta)}$ be the free $\Bbbk$-vector space on $\mathscr{F}_i(\Delta)$. The *(reduced) chain complex* of M over $\Bbbk$ is the complex

$$0 \leftarrow \Bbbk^{\mathscr{F}_{-1}(\Delta)} \xleftarrow{\delta_0} \cdots \leftarrow \Bbbk^{\mathscr{F}_{i-1}(\Delta)} \xleftarrow{\delta_i} \Bbbk^{\mathscr{F}_i(\Delta)} \leftarrow \cdots \xleftarrow{\delta_{\dim(\Delta)}} \Bbbk^{\mathscr{F}_{\dim(\Delta)}(\Delta)} \leftarrow 0,$$

where the boundary maps δ_i are defined as follows: With the natural ordering on E, set $\operatorname{sign}(j,\sigma) = (-1)^{r-1}$ if j is the r^{th} element of $\sigma \subseteq E$, and let

$$\delta_i(\sigma) = \sum_{j \in \sigma} \operatorname{sign}(j,\sigma)\; \sigma \smallsetminus j.$$

Extending δ_i $\Bbbk$-linearly, we obtain a $\Bbbk$-linear map from $\Bbbk^{\mathscr{F}_i(\Delta)}$ to $\Bbbk^{\mathscr{F}_{i-1}(\Delta)}$.

Definition 2.14. The i^{th} *reduced homology* of Δ over $\Bbbk$ is the vector space

$$\tilde{H}_i(\Delta;\Bbbk) = \ker(\delta_i)/\operatorname{im}(\delta_{i+1}).$$

The following is one of the most celebrated results in the intersection between algebra and combinatorics.

Theorem 2.2 (Hochster's formula)**.**

$$\beta_{i,\sigma}(S/I_\Delta;\Bbbk) = \beta_{i-1,\sigma}(I_\Delta;\Bbbk) = \dim_\Bbbk \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma};\Bbbk).$$

Proof. See [11, Corollary 5.12] and [7, p. 81]. □

3 Betti numbers of *i*-skeletons

Let Δ be a d-dimensional simplicial complex on $\{1,\dots,n\}$, and let $\Bbbk$ be a field. In this section we shall demonstrate how each of the Betti numbers of $S/I_{\Delta^{(d-1)}}$ can be expressed as a $\mathbb{Z}$-linear combination of the Betti numbers of S/I_Δ.

3.1 The first rows of the Betti table

Lemma 3.1.
$$\tilde{H}_i(\Delta_{|\sigma};\Bbbk) = \tilde{H}_i(\Delta^{(d-1)}{}_{|\sigma};\Bbbk)$$
for all $0 \le i \le d-2$.

Proof. By the definition of a skeleton we have $\mathscr{F}_i(\Delta_{|\sigma}) = \mathscr{F}_i(\Delta^{(d-1)}{}_{|\sigma})$ and thus also $\Bbbk^{\mathscr{F}_i(\Delta_{|\sigma})} = \Bbbk^{\mathscr{F}_i(\Delta^{(d-1)}{}_{|\sigma})}$, for all $-1 \le i \le d-1$. In other words, the reduced chain complexes of $\Delta_{|\sigma}$ and $\Delta^{(d-1)}{}_{|\sigma}$ are identical except for in homological degree d. The result follows. □

Proposition 3.1. *For all i and $j \le d+i-1$ we have*

$$\beta_{i,j}(S/I_\Delta;\Bbbk) = \beta_{i,j}(S/I_{\Delta^{(d-1)}};\Bbbk).$$

Proof. If $j \leq d+i-1$ then $j-i-1 \leq d-2$. By Theorem 2.2 and Lemma 3.1 then, we have

$$\begin{aligned}
\beta_{i,j}(S/I_\Delta;\Bbbk) &= \sum_{|\sigma|=j} \beta_{i,\sigma}(S/I_\Delta;\Bbbk) \\
&= \sum_{|\sigma|=j} \dim_\Bbbk \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma};\Bbbk) \\
&= \sum_{|\sigma|=j} \dim_\Bbbk \tilde{H}_{|\sigma|-i-1}({\Delta^{(d-1)}}_{|\sigma};\Bbbk) \\
&= \sum_{|\sigma|=j} \beta_{i,\sigma}(S/I_{\Delta^{(d-1)}};\Bbbk) \\
&= \beta_{i,j}(S/I_{\Delta^{(d-1)}};\Bbbk).
\end{aligned}$$

□

3.2 The final row of the Betti table

The Hilbert series of S/I_Δ over $\Bbbk$ is $H(S/I_\Delta) = \sum_{i\in\mathbb{Z}} \dim_\Bbbk (S/I_\Delta)_i\, t^i$. Let $f_i(\Delta) = |\mathscr{F}_i(\Delta)|$. By [7, Section 6.1.3, Equation (6.3)] we have

$$H(S/I_\Delta) = \frac{\sum_{i=0}^{n}(-1)^i \sum_j \beta_{i,j}(S/I_\Delta;\Bbbk)}{(1-t)^n}.$$

On the other hand, we see from [7, Proposition 6.2.1] that

$$H(S/I_\Delta) = \frac{\sum_{i=0}^{d+1} f_{i-1}(\Delta)t^i(1-t)^{d+1-i}}{(1-t)^{d+1}}.$$

Combined, these two equations imply

$$\sum_{i=0}^{d+1} f_{i-1}(\Delta)t^i(1-t)^{n-i} = \sum_{i=0}^{n}(-1)^i \sum_j \beta_{i,j}(S/I_\Delta;\Bbbk)t^j, \tag{2}$$

and

$$\sum_{i=0}^{d} f_{i-1}(\Delta^{(d-1)})t^i(1-t)^{n-i} = \sum_{i=0}^{n}(-1)^i \sum_j \beta_{i,j}(S/I_{\Delta^{(d-1)}};\Bbbk)t^j. \tag{3}$$

Remark. From here on we shall employ the convention that $i! = 0$ for $i < 0$, and that $\binom{j}{k} = 0$ if one or both of j and k is negative.

Differentiating both sides of equation (2) $n-d-1$ times, we get

$$\sum_{i=0}^{d+1} f_{i-1}(\Delta) \sum_{l=0}^{n-d-1} (-1)^l \binom{n-d-1}{l} \frac{i!(n-i)!}{(i-n+d+1+l)!(n-i-l)!} t^{i-n+d+1+l}(1-t)^{n-i-l}$$
$$= \sum_{i=0}^{n} (-1)^i \sum_j \beta_{i,j}(S/I_\Delta;\Bbbk) \frac{j!}{(j-(n-d-1))!} t^{j-n+d+1}.$$

When evaluated at $t=1$, the left side of the above equation is 0 except when $i=d+1$ and $l=n-d-1$. Thus, we have

$$(-1)^{n-d-1}(n-d-1)!\, f_d(\Delta) = \sum_{i=0}^{n} (-1)^i \sum_{j\geq n-d-1} \beta_{i,j}(S/I_\Delta;\Bbbk) \frac{j!}{(j-(n-d-1))!},$$

and

$$f_d(\Delta) = \sum_{i=0}^{n} (-1)^{n+d+i+1} \sum_{j\geq n-d-1} \binom{j}{n-d-1} \beta_{i,j}(S/I_\Delta;\Bbbk).$$

Lemma 3.2. *For all i and $j \geq d+i+2$ we have*

$$\beta_{i,j}(S/I_\Delta;\Bbbk) = 0.$$

Proof. If $|\sigma| \geq d+i+2$, then $|\sigma|-i-1 \geq \dim(\Delta)+1$, which implies

$$\dim_\Bbbk \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma};\Bbbk) = 0.$$

So by Hochster's formula we have that if $j \geq d+i+2$ then

$$\beta_{i,j}(S/I_\Delta;\Bbbk) = \sum_{|\sigma|=j} \beta_{i,\sigma}(S/I_\Delta;\Bbbk) = \sum_{|\sigma|=j} \dim_\Bbbk \tilde{H}_{|\sigma|-i-1}(\Delta_{|\sigma};\Bbbk) = 0.$$

□

According to Proposition 3.1 and Lemma 3.2, and because $f_i(\Delta) = f_i(\Delta^{(d-1)})$ for all $i \neq d$, subtracting equation (3) from equation (2) yields

$$f_d(\Delta) t^{d+1}(1-t)^{n-d-1} = \sum_{i=0}^{n} (-1)^i \big(\beta_{i,d+i}(S/I_\Delta;\Bbbk) - \beta_{i,d+i}(S/I_{\Delta^{(d-1)}};\Bbbk)\big) t^{d+i}$$
$$+ \sum_{i=0}^{n} (-1)^i \beta_{i,d+i+1}(S/I_\Delta;\Bbbk) t^{d+i+1}.$$

Let $1 \leq u \leq n$. Differentiating both sides of the above equation $d+u$ times yields

$$
\begin{aligned}
& f_d(\Delta)\sum_{l=0}^{d+u}(-1)^l\binom{d+u}{l}\frac{(d+1)!(n-d-1)!}{(l-u+1)!(n-d-1-l)!}t^{l-u+1}(1-t)^{n-d-1-l}\\
=& \sum_{i=u}^{n}(-1)^i\big(\beta_{i,d+i}(S/I_\Delta;\Bbbk)-\beta_{i,d+i+1}(S/I_{\Delta^{(d-1)}};\Bbbk)\big)\frac{(d+i)!}{(i-u)!}t^{i-u}\\
& +\sum_{i=u-1}^{n}(-1)^i\beta_{i,d+i+1}(S/I_\Delta;\Bbbk)\frac{(d+i+1)!}{(i-u+1)!}t^{i-u+1}.
\end{aligned}
$$

Evaluating at $t=0$, we get

$$
\begin{aligned}
& \delta' * \left((-1)^{u-1}f_d(\Delta)\frac{(d+u)!(n-d-1)!}{(u-1)!(n-d-u)!}\right)\\
=&(-1)^u\ big(\beta_{u,d+u}(S/I_\Delta;\Bbbk)-\beta_{u,d+u}(S/I_{\Delta^{(d-1)}};\Bbbk)\big)(d+u)!\\
& +(-1)^{u-1}\beta_{u-1,d+u}(S/I_\Delta;\Bbbk)(d+u)!,
\end{aligned}
$$

where

$$
\delta' = \begin{cases} 1, & 1 \leq u \leq n-d \\ 0, & u > n-d \end{cases}.
$$

Summarizing the above:

Proposition 3.2. *For $1 \leq u \leq n$, we have*

$$
\beta_{u,d+u}(S/I_{\Delta^{(d-1)}};\Bbbk) = \beta_{u,d+u}(S/I_\Delta;\Bbbk) - \beta_{u-1,d+u}(S/I_\Delta;\Bbbk) + \binom{n-d-1}{u-1}\delta,
$$

where

$$
\delta = \begin{cases} f_d(\Delta) = \sum_{i=0}^{n}(-1)^{n+d+i+1}\sum_{j\geq n-d-1}\binom{j}{n-d-1}\beta_{i,j}(S/I_\Delta;\Bbbk), & 1 \leq u \leq n-d \\ 0, & u > n-d. \end{cases}
$$

Bringing together Propositions 3.1 and 3.2, we get

Theorem 3.1. *For all $i \geq 1$, we have*

$$
\beta_{i,j}(S/I_{\Delta^{(d-1)}};\Bbbk) = \begin{cases} \beta_{i,j}(S/I_\Delta;\Bbbk), & j \leq d+i-1 \\ \beta_{i,d+i}(S/I_\Delta;\Bbbk) - \beta_{i-1,d+i}(S/I_\Delta;\Bbbk) + \binom{n-d-1}{i-1}\delta, & j = d+i, \\ 0, & j \geq d+i-1 \end{cases}
$$

where

$$\delta = \begin{cases} f_d(\Delta) = \sum_{k=0}^{n}(-1)^{n+d+k+1}\sum_{j\geq n-d-1}\binom{j}{n-d-1}\beta_{k,j}(S/I_\Delta;\mathbb{k}), & 1\leq i\leq n-d \\ 0, & i > n-d. \end{cases}$$

Example 3.1. Let T be one of the two irreducible triangulations of the real projective plane (see [1]) – namely the one corresponding to an embedding of the complete graph on 6 vertices. Clearly then, we have $n = 6$ and $d = 2$. The Betti table of S/I_T over $\mathbb{F}_3$ is

$$\beta[S/I_T](\mathbb{F}_3) = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 3 & 0 & 10 & 15 & 6 \end{array}.$$

In this case $f_d(\Delta) = \binom{4}{3}\beta_{1,4}(S/I_T;\mathbb{F}_3) - \binom{5}{3}\beta_{2,5}(S/I_T;\mathbb{F}_3) + \binom{6}{3}\beta_{3,6}(S/I_T;\mathbb{F}_3) = 10$. By Theorem 3.1, the Betti numbers of $S/I_{T^{(1)}}$ are

$$\begin{aligned}
\beta_{1,4}(S/I_{T^{(1)}};\mathbb{F}_3) =& \beta_{1,4}(S/I_T;\mathbb{F}_3) + \binom{3}{0}\delta = 10+10. \\
\beta_{2,5}(S/I_{T^{(1)}};\mathbb{F}_3) =& \beta_{2,5}(S/I_T;\mathbb{F}_3) - \beta_{1,5}(S/I_T;\mathbb{F}_3) + \binom{3}{1}\delta = 15+30. \\
\beta_{3,6}(S/I_{T^{(1)}};\mathbb{F}_3) =& \beta_{3,6}(S/I_T;\mathbb{F}_3) - \beta_{2,6}(S/I_T;\mathbb{F}_3) + \binom{3}{2}\delta = 6-0+30. \\
\beta_{4,7}(S/I_{T^{(1)}};\mathbb{F}_3) =& \beta_{4,7}(S/I_T;\mathbb{F}_3) - \beta_{3,7}(S/I_T;\mathbb{F}_3) + \binom{3}{3}\delta = 0-0+10.
\end{aligned}$$

$$\beta[S/I_{T^{(1)}}](\mathbb{F}_3) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 20 & 45 & 36 & 10 \end{array}.$$

Remark. Observe that as

$$\beta[S/I_T](\mathbb{F}_2) = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 10 & 15 & 6 & 1 \\ 4 & 0 & 0 & 0 & 1 & 0 \end{array},$$

the simplicial complex T of Example 3.1 is an example of a pure simplicial complex whose Betti numbers depend upon the field $\Bbbk$ – as opposed to what is the case for matroids.

3.3 The projective dimension of skeletons

Let $\mathrm{p.d.}\, S/I_\Delta$ denote the projective dimension of S/I_Δ. By Auslander-Buchsbaum Theorem we have

$$\begin{aligned} \mathrm{p.d.}\, S/I_\Delta =& n - \operatorname{depth} S/I_\Delta \\ \geq& n - \dim S/I_\Delta \\ =& n - (d+1), \end{aligned}$$

so $n-d-1 \leq \mathrm{p.d.}\, S/I_\Delta \leq n$.

As for the skeletons, we have

Corollary 3.1.

$$\mathrm{p.d.}\, S/I_{\Delta^{(d-1)}} \leq 1 + \mathrm{p.d.}\, S/I_\Delta.$$

Proof. Let $p = \mathrm{p.d.}\, S/I_\Delta$. By Proposition 3.1 it suffices to show that

$$\beta_{p+2,d+p+2}(S/I_{\Delta^{(d-1)}};\Bbbk) = 0.$$

But by Theorem 3.2, we have

$$\begin{aligned} \beta_{p+2,d+p+2}(S/I_{\Delta^{(d-1)}};\Bbbk) =& \beta_{p+2,d+p+2}(S/I_\Delta;\Bbbk) - \beta_{p+1,d+p+2}(S/I_\Delta;\Bbbk) + \delta \\ =& 0 - 0 - \delta = 0, \end{aligned}$$

where the last equality is due to $p+2 > n-d$. □

Corollary 3.2. *If Δ is Cohen-Macauley, then so is $\Delta^{(d-1)}$.*

Proof. Let Δ be a simplicial complex with $\dim(\Delta) = d$ and depth $S/I_\Delta = \dim S/I_\Delta$. As $\dim S/I_{\Delta^{(d-1)}} = d$, we only need to prove that depth $S/I_{\Delta^{(d-1)}} = d$ as well.

Since depth $S/I_{\Delta^{(d-1)}} \leq \dim S/I_{\Delta^{(d-1)}} = d$, we have by the Auslander-Buchsbaum Theorem that p. d. $S/I_{\Delta^{(d-1)}} \geq n-d$. On the other hand, since

$$\begin{aligned} \text{p. d. } S/I_\Delta =& n - \text{depth } S/I_\Delta \\ =& n - \dim S/I_\Delta \\ =& n-(d+1), \end{aligned}$$

we see from Corollary 3.1 that p. d. $S/I_{\Delta^{(d-1)}} \leq n-d$. We conclude that

$$\text{p. d. } S/I_{\Delta^{(d-1)}} = n-d$$

and, by Auslander-Buchsbaum again, that depth $S/I_{\Delta^{(d-1)}} = d$. □

4 Betti numbers of truncations and elongations of matroids

Let M be a matroid on $\{1,\dots,n\}$, with $r(M) = k$. As was established in [3], the dimension of $\tilde{H}_i(M;\Bbbk)$ is in fact independent of the field $\Bbbk$. Thus *for matroids, the ($\mathbb{N}_0$- or $\mathbb{N}_0^n$-graded) Betti numbers are not only unique, but independent of the choice of field*. We shall therefore omit referring to or specifying a particular field $\Bbbk$ throughout this section. By a slight abuse of notation we shall denote the Stanley-Reisner ideal associated to the set of independent sets $I(M)$ of M simply by I_M.

4.1 Truncations

Note that the i^{th} truncation of M corresponds to the $(k-i-1)$-skeleton of $I(M)$; a fact which enables us to invoke Theorem 3.1. In addition, it follows from [9, Corollary 3(b)] that the minimal free resolutions of S/I_M have length $n-k$. We thus have

Proposition 4.1. *For all i, we have*

$$\beta_{i,j}(S/I_{M^{(1)}}) = \begin{cases} \beta_{i,j}(S/I_M), & j \leq k+i-2. \\ \beta_{i,k+i-1}(S/I_M) - \beta_{i-1,k+i-1}(S/I_M) & \\ +\binom{n-k}{i-1}\left(\sum_{u=0}^{n-k}(-1)^{n+k+u}\sum_{v\geq n-k}\binom{v}{n-k}\beta_{u,v}(S/I_M)\right), & j = k+i-1. \\ 0, & j \geq k+i. \end{cases}$$

4.2 Elongations

When it comes to elongations, the Betti numbers of M provide far less information about the Betti numbers of $M_{(1)}$ than what was the case with truncations. We do however have the following.

Proposition 4.2. *For $i \geq 1$,*

$$\beta_{i,j}(I_{M_{(l)}}) \neq 0 \iff \beta_{i-1,j}(I_{M_{(l+1)}}) \neq 0.$$

Proof. According to [9, Theorem 1], we have that

$$\beta_{i,\sigma}(I_M) \neq 0 \iff \sigma \text{ is minimal with the property that } n_M(\sigma) = i+1.$$

Since $\beta_{i,j} = \sum_{|\sigma|=j} \beta_{i,\sigma}$, we see that

$$\beta_{i,j}(I_{M_{(l)}}) \neq 0$$

$$\iff$$

There is a σ such that $|\sigma| = j$ and σ is minimal with the property that $n_{M_{(l)}}(\sigma) = i+1$

$$\iff$$

There is a σ such that $|\sigma| = j$ and σ is minimal with the property that $n_{M_{(l+1)}}(\sigma) = i$

$$\iff$$

$$\beta_{i-1,j}(I_{M_{(l+1)}}) \neq 0.$$

□

In terms of Betti tables, this implies that when it comes to zeros and nonzeros the Betti table of $I_{M_{(i+1)}}$ is equal to the table you get by deleting the first column from the table of I_{M_i}. As the following counterexample (computed using MAGMA [2]) demonstrates, there can be no result for elongations analogous to Theorem 3.1.

Let M and N be the matroids on $\{1,\dots,8\}$ with bases

$$\begin{aligned} B(M) = \big\{ &\{1,3,4,6,7\},\{1,2,3,6,8\},\{1,2,3,4,8\},\{1,2,3,5,8\},\{1,2,5,6,8\},\\ &\{1,2,3,4,7\},\{1,2,3,5,7\},\{1,2,5,6,7\},\{1,3,4,5,7\},\{1,3,4,6,8\},\\ &\{1,2,4,6,8\},\{1,2,4,6,7\},\{1,3,4,5,8\},\{1,2,4,5,7\},\{1,4,5,6,7\},\\ &\{1,2,3,6,7\},\{1,3,5,6,7\},\{1,4,5,6,8\},\{1,3,5,6,8\},\{1,2,4,5,8\}\big\} \end{aligned}$$

and

$$
\begin{aligned}
B(N) = \big\{&\{1,3,4,6,7\},\{1,2,3,4,8\},\{1,2,3,5,8\},\{1,2,5,6,8\},\{1,2,3,4,7\},\\
&\{1,2,3,5,7\},\{1,2,5,6,7\},\{1,3,4,5,7\},\{1,3,4,6,8\},\{1,2,4,6,8\},\\
&\{1,2,4,6,7\},\{1,3,4,5,8\},\{1,2,4,5,7\},\{1,3,4,5,6\},\{1,2,4,5,6\},\\
&\{1,3,5,6,7\},\{1,2,3,5,6\},\{1,2,3,4,6\},\{1,3,5,6,8\},\{1,2,4,5,8\}\big\}.
\end{aligned}
$$

Both I_M and I_N have Betti table

$$
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\
\hline
2 & 1 & 0 & 0 \\
3 & 0 & 0 & 0 \\
4 & 1 & 4 & 0 \\
5 & 0 & 5 & 4
\end{array},
$$

but while $I_{M_{(1)}}$ has Betti table

$$
\begin{array}{c|cc}
 & 1 & 2 \\
\hline
5 & 1 & 0 \\
6 & 5 & 5
\end{array}
$$

the ideal $I_{N_{(1)}}$ has Betti table

$$
\begin{array}{c|cc}
 & 1 & 2 \\
\hline
5 & 2 & 0 \\
6 & 3 & 4
\end{array}.
$$

This shows that the Betti numbers associated to a matroid do not determine those associated to its elongation.

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