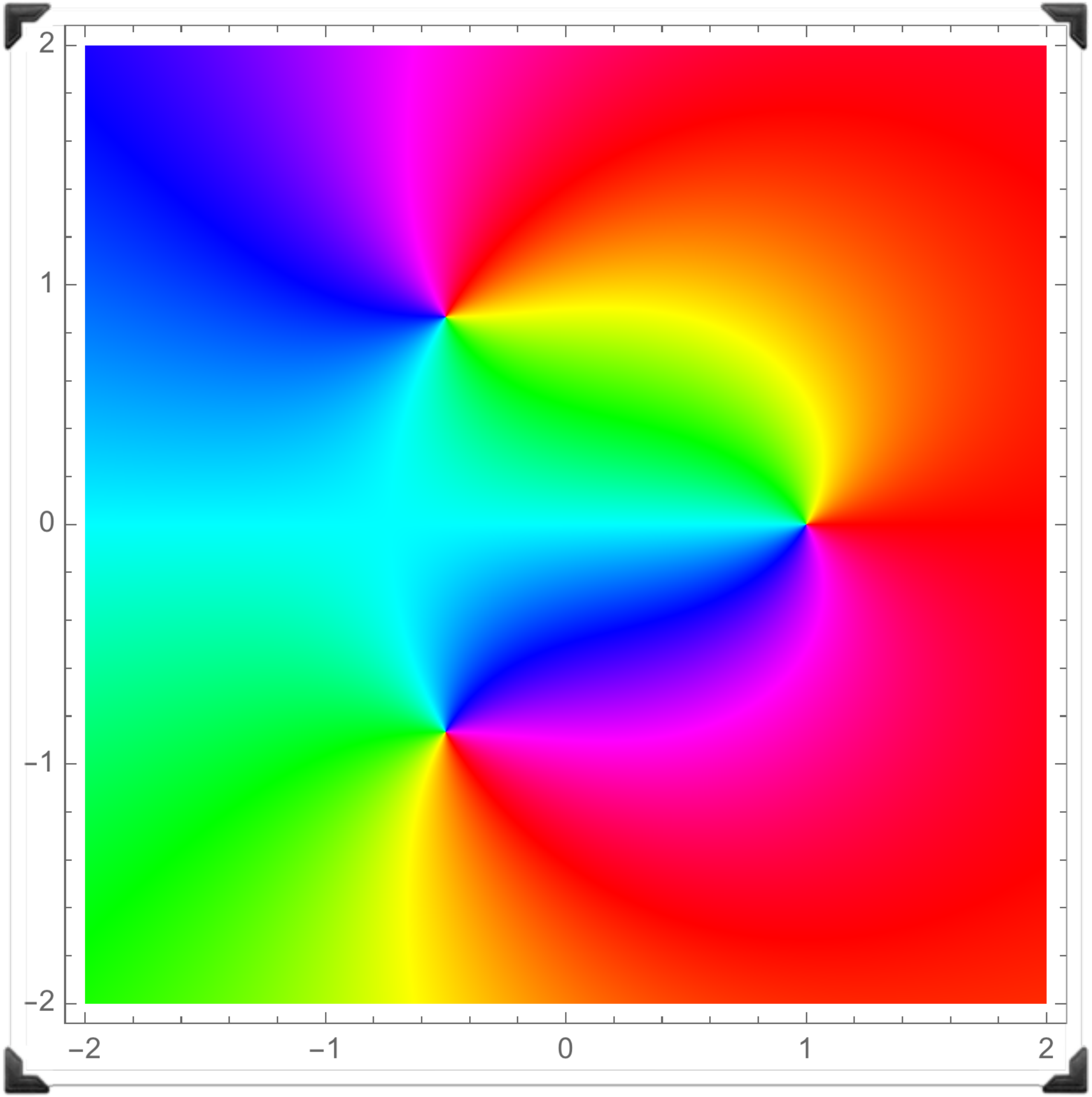
COMPLEX BEAUTIES

2017

Let's play a game!

I will show you the **phase plot** of a holomorphic function…

e.g.,



and you tell me which function it is!

What is a phase plot?

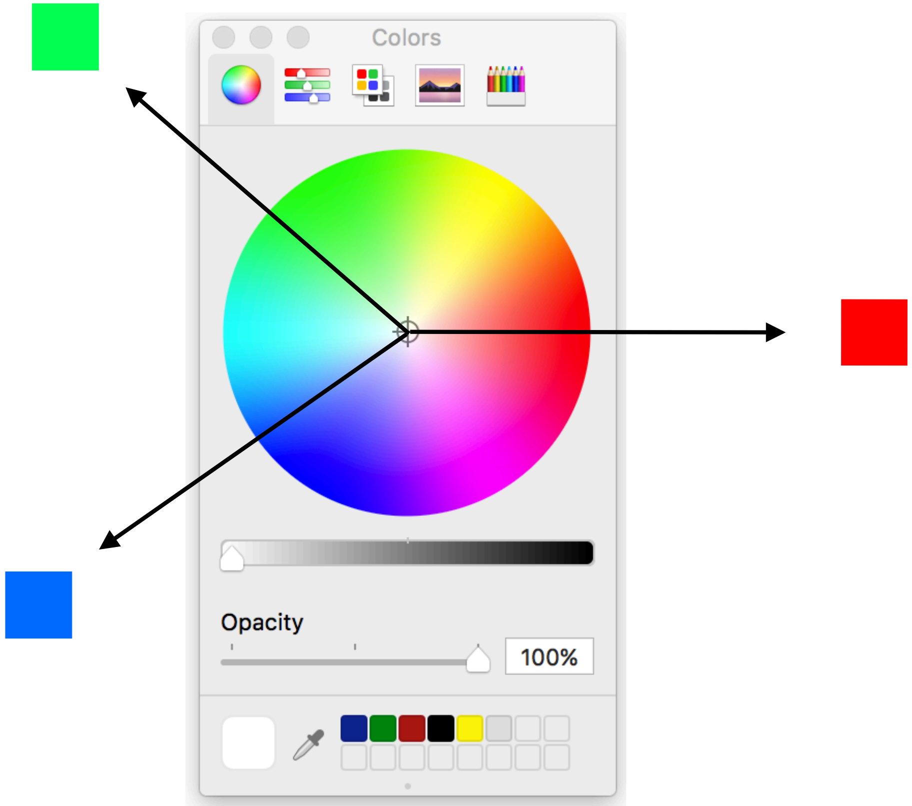
Consider a holomorphic function

$$w = f(z)$$

Its phase plot is a colouring of the complex plane, where the point z is coloured according to the value of the argument of w

To do that we use a colour wheel!

The rays of the colour wheel have the same “hue”, so we map the argument to the hue.



You can recover the holomorphic function (up to a real **positive** scaling) from its phase plot!

(Think about it!)

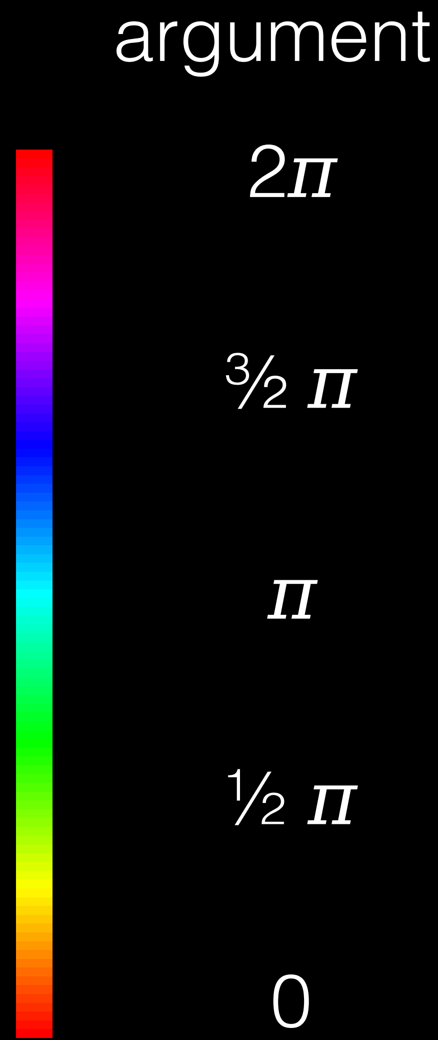
An example

$$w = f(z) = z$$

This shows the colour scheme:

green to blue:
increasing argument

blue to green:
decreasing argument



Recall the **argument principle**…

winding number about the origin

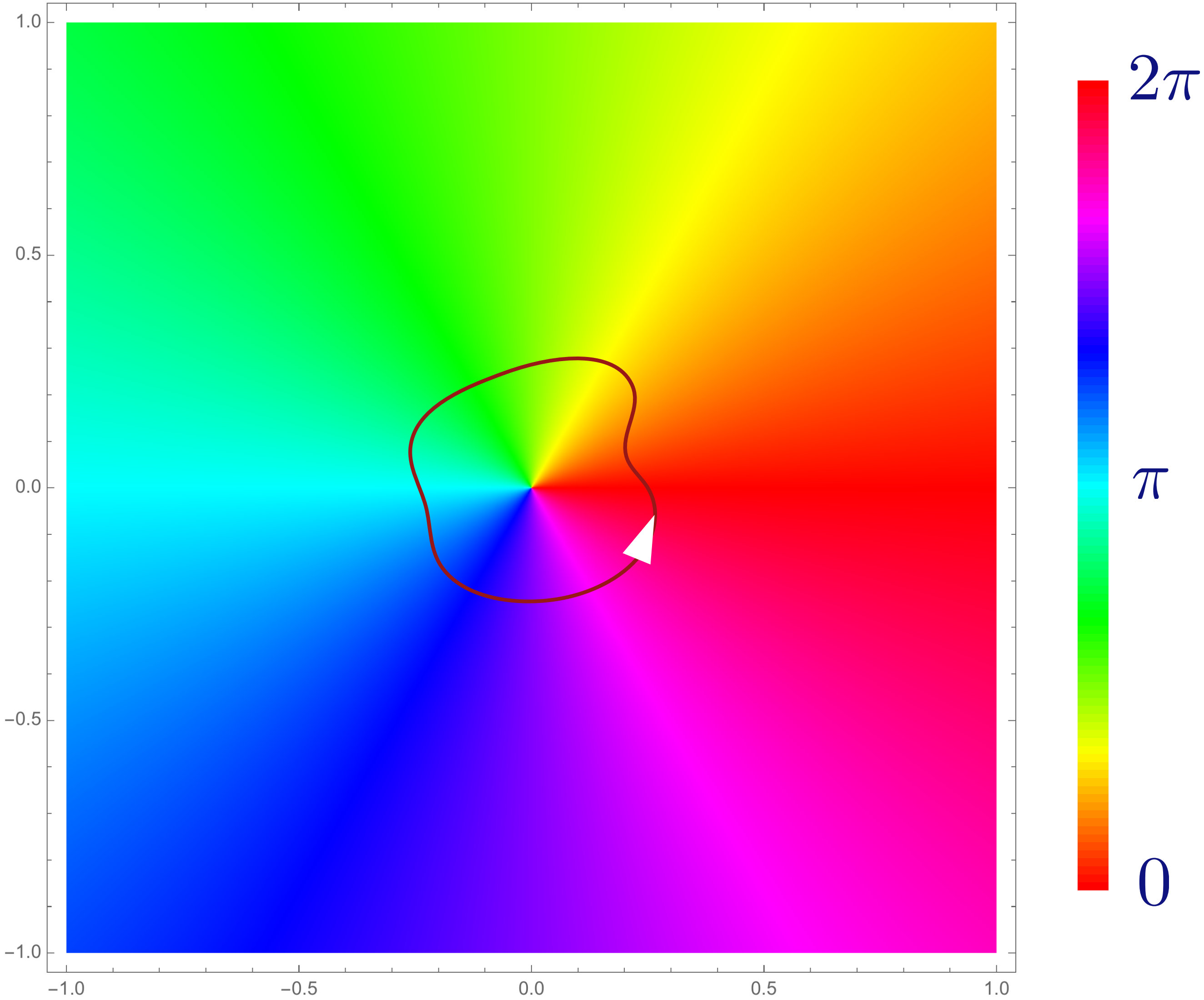
=

number of zeros - number of poles

We can determine the winding number by counting how many times the colours appear.

$$w = f(z) = z$$

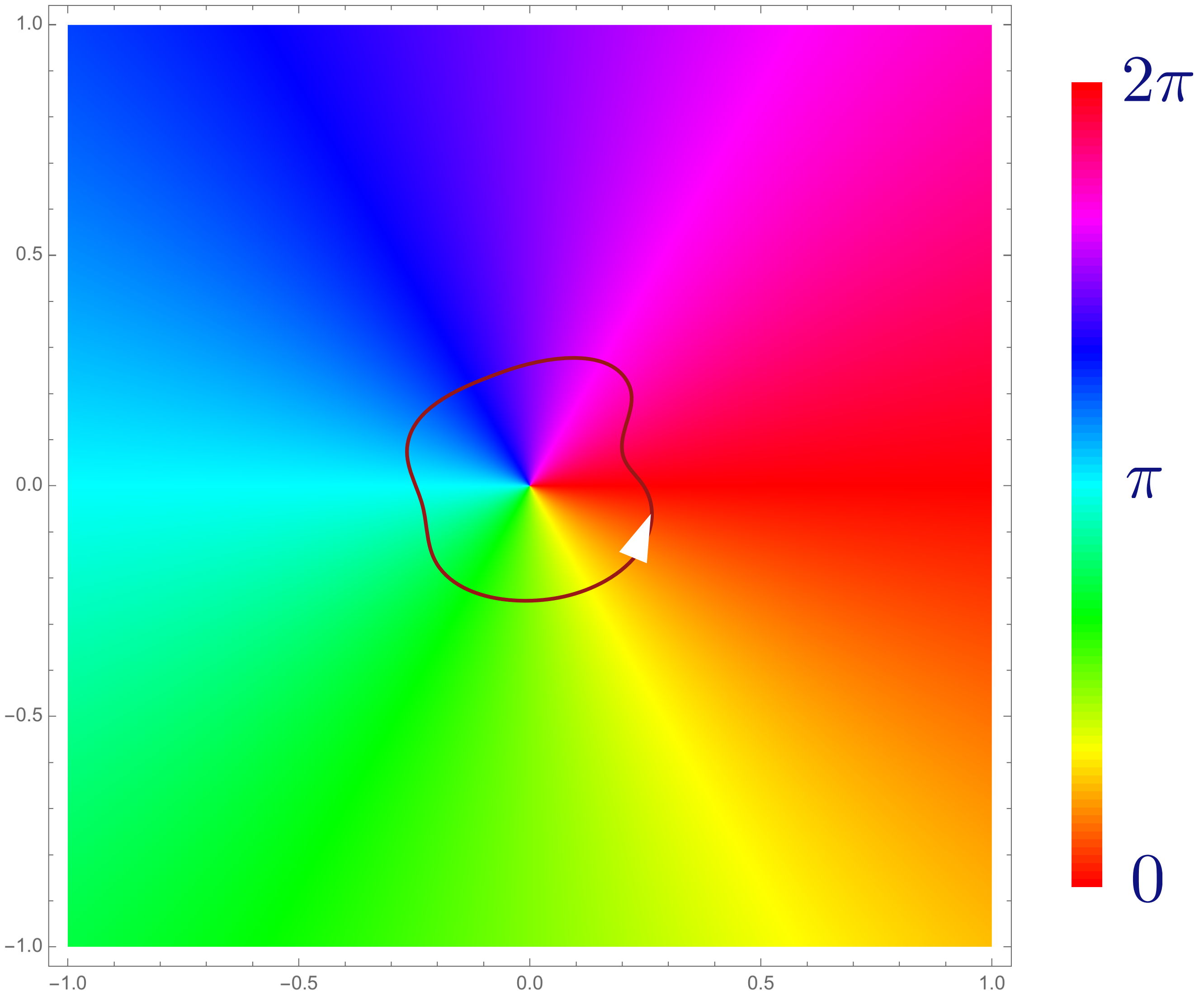
Going around a (positively-oriented) loop, we go around the colour wheel once in the **positive** sense.



Argument Principle: ∃ simple **zero** inside the loop.

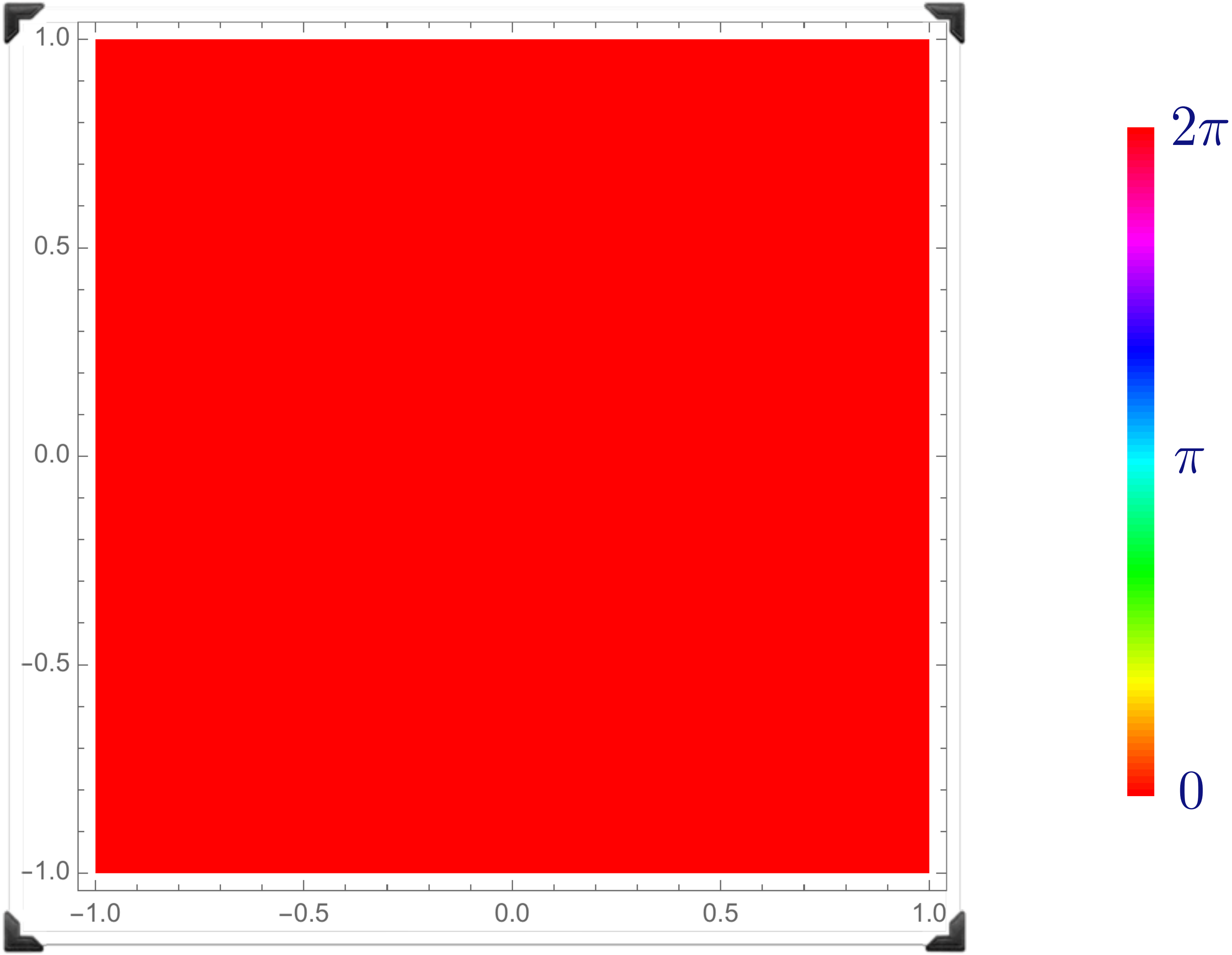
$$w = f(z) = \frac{1}{z}$$

Going around the loop, we go around the colour wheel once in the **negative** sense.



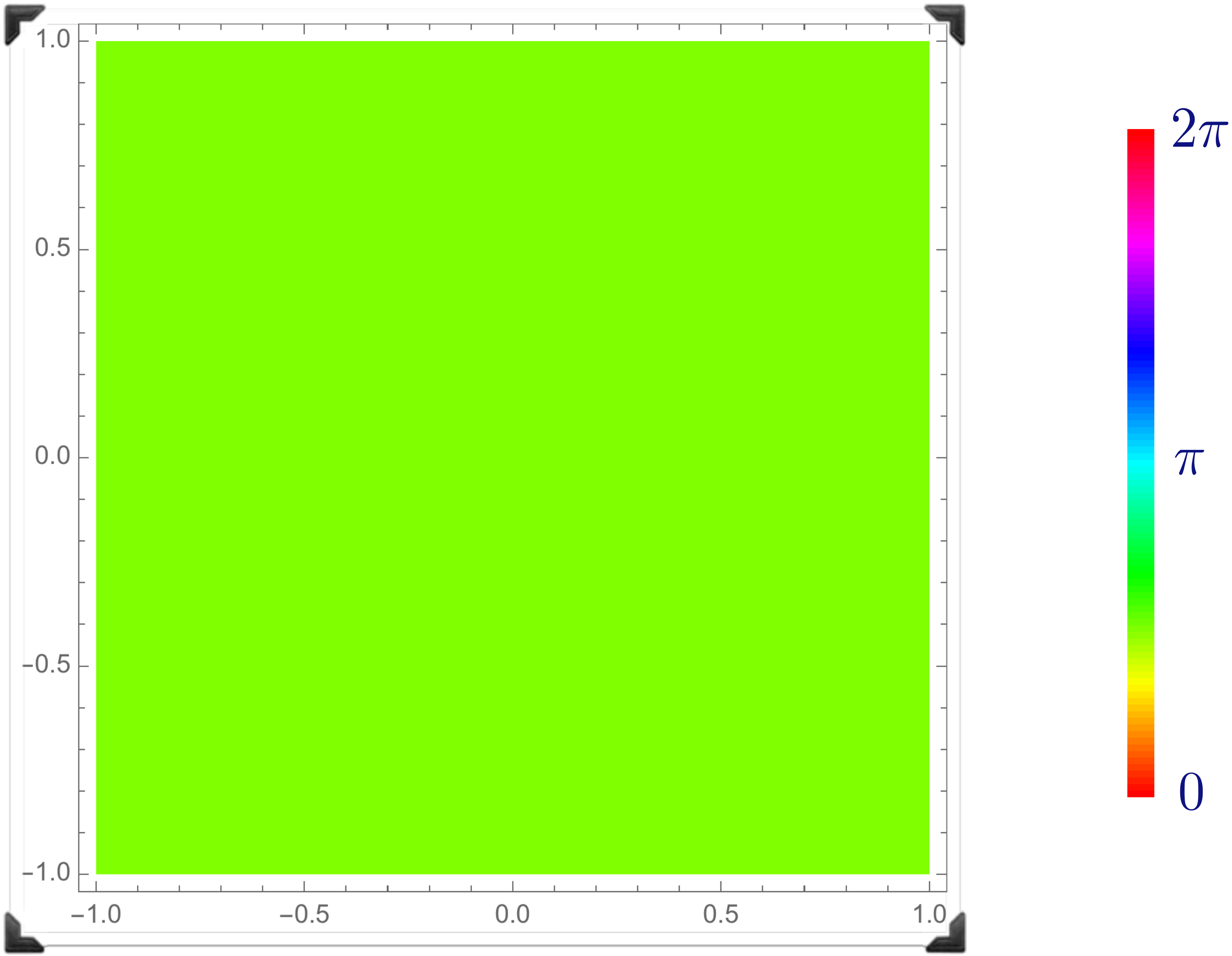
Argument Principle: ∃ simple **pole** inside the loop.

Ready?


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

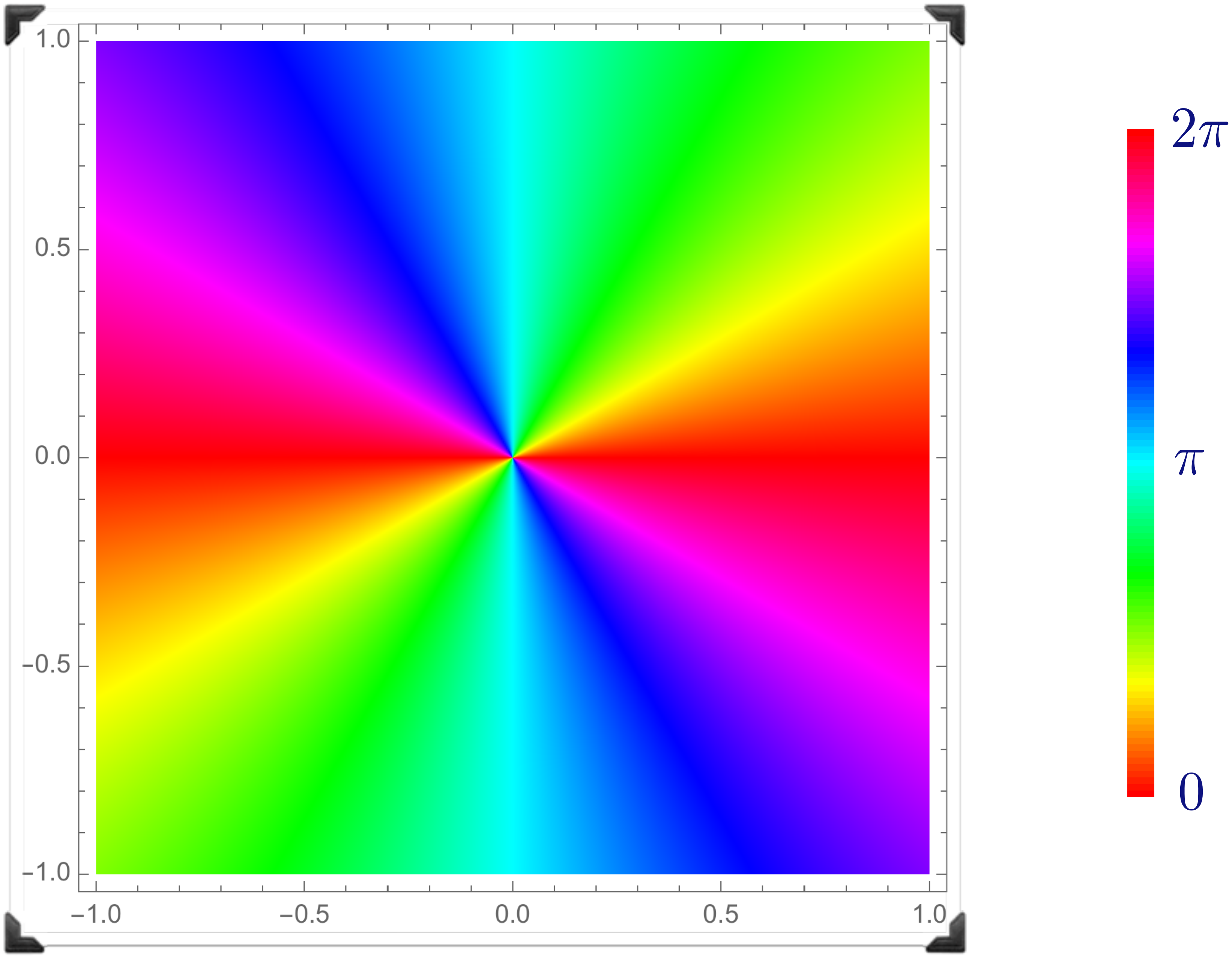
?

$$w = f(z) = 1$$


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

?

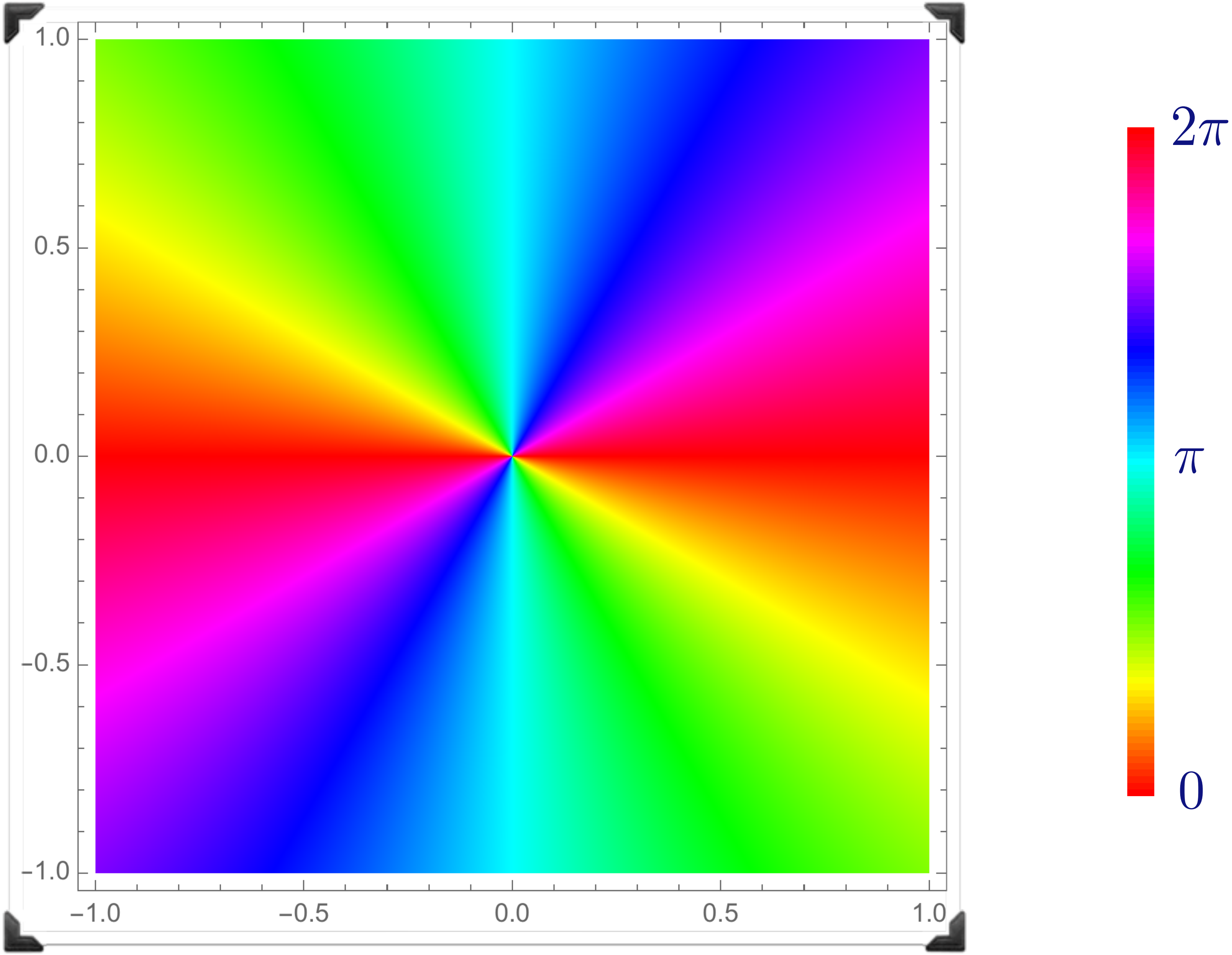
$$w = f(z) = i$$



1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

?

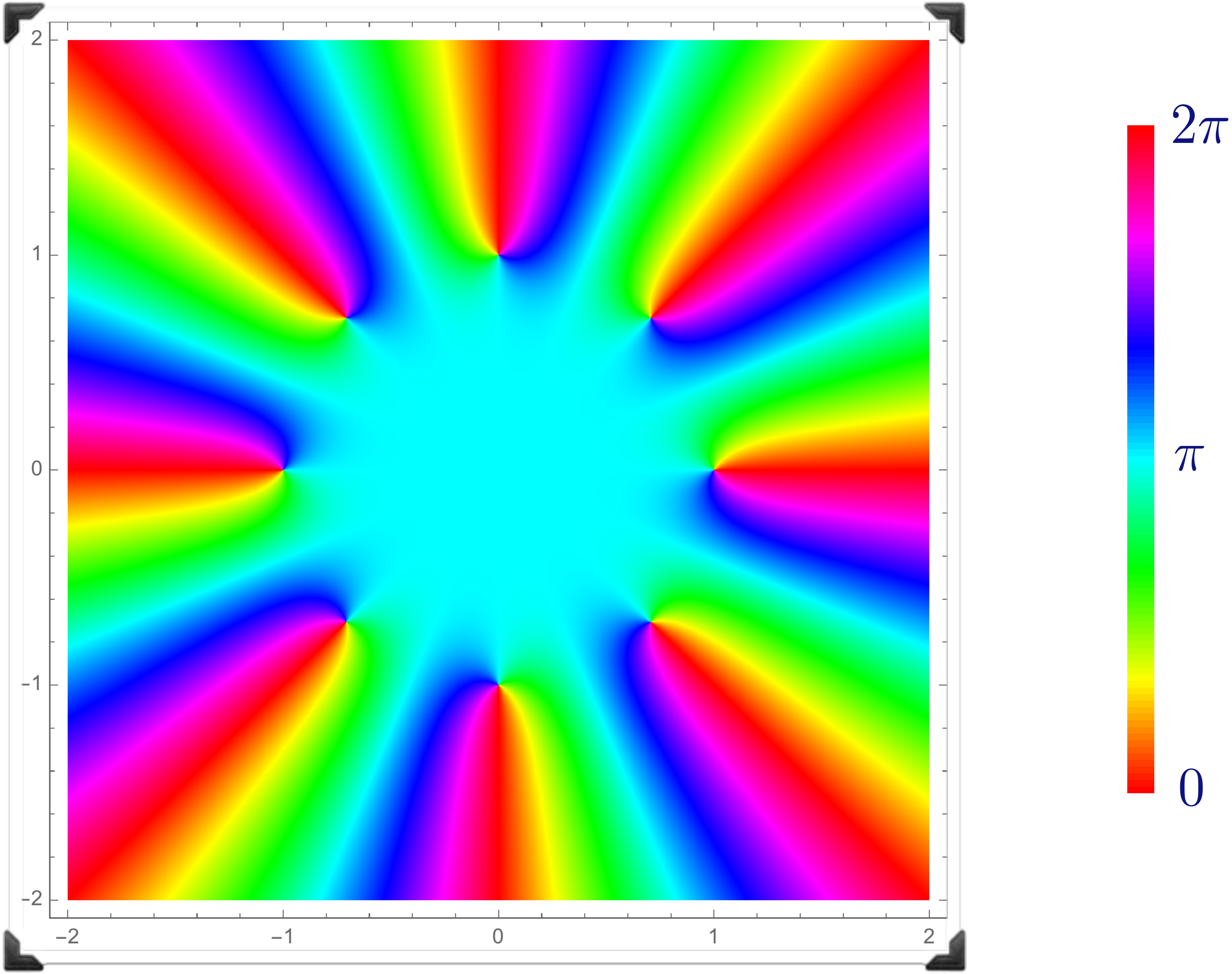
$$w = f(z) = z^2$$



1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

?

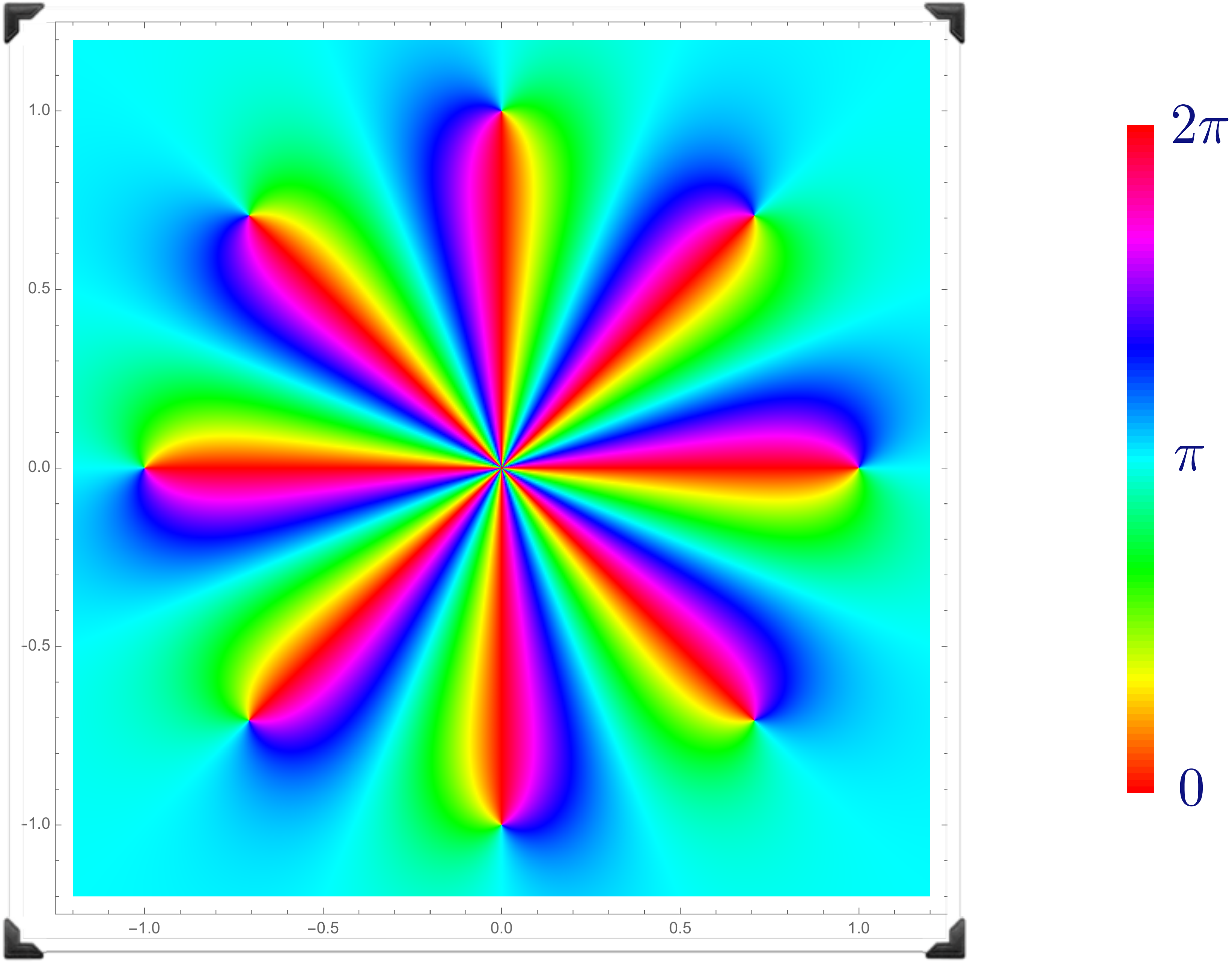
$$w = f(z) = \frac{1}{z^2}$$



2
1
0
-1
-2
-2
-1
0
1
2
2π
π
0

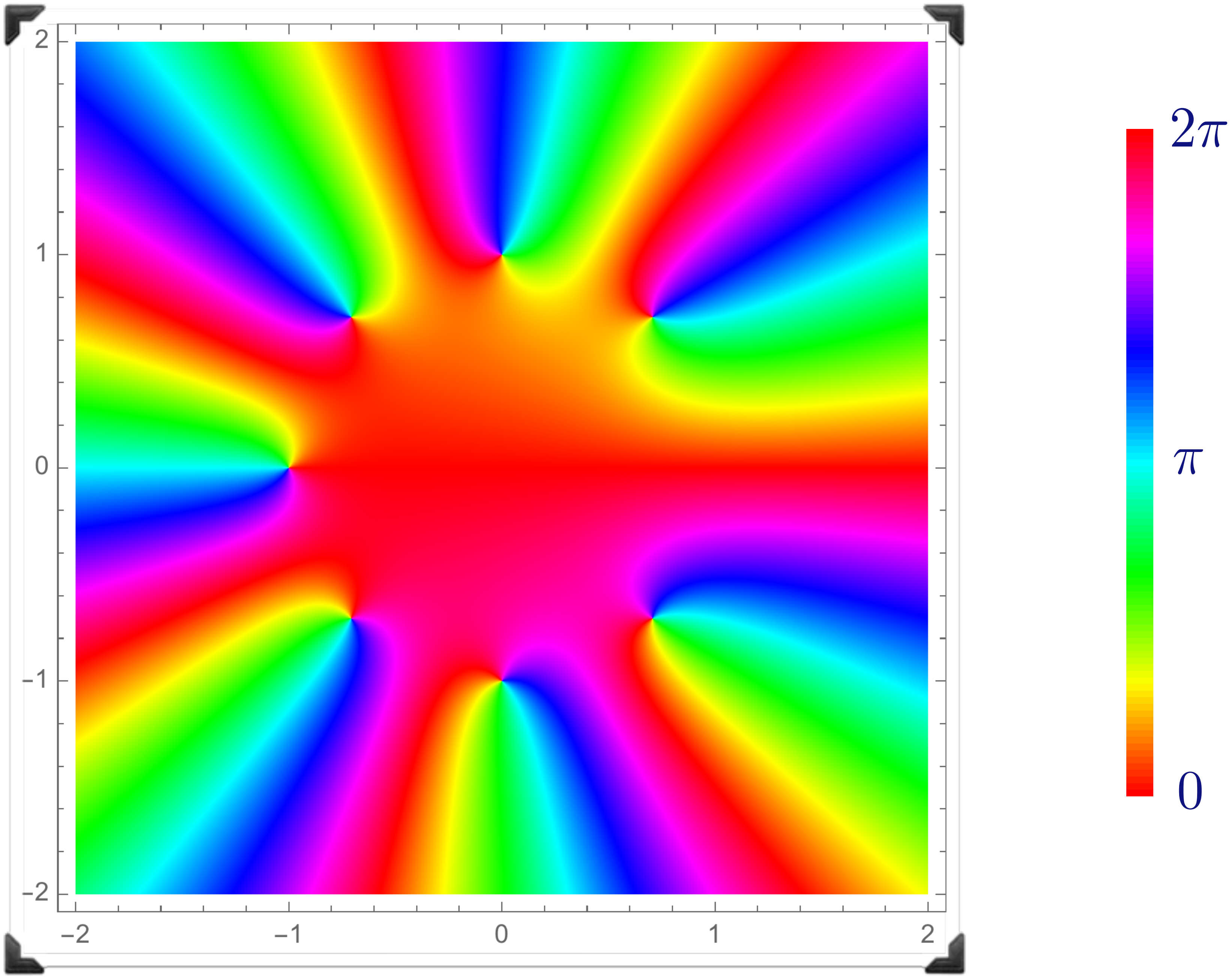
?

$$w = f(z) = z^8 - 1$$


1.0
0.5
0.0
-0.5
-1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

?

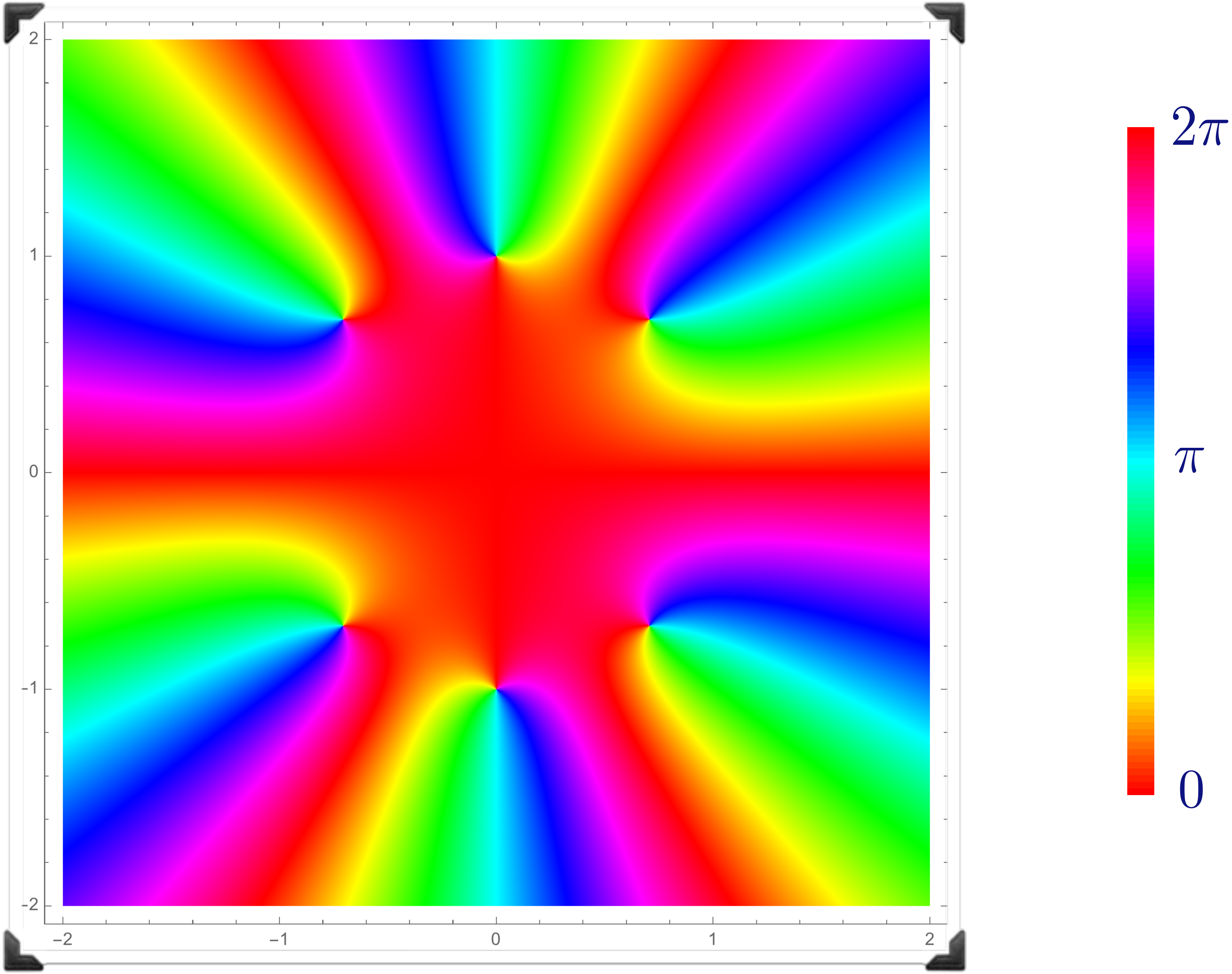
$$
\begin{aligned}
w = f(z) &= \frac{1}{z^8} - 1 \\
&= \frac{1 - z^8}{z^8}
\end{aligned}
$$



2
1
0
−1
−2
−2
−1
0
1
2
2π
π
0

?

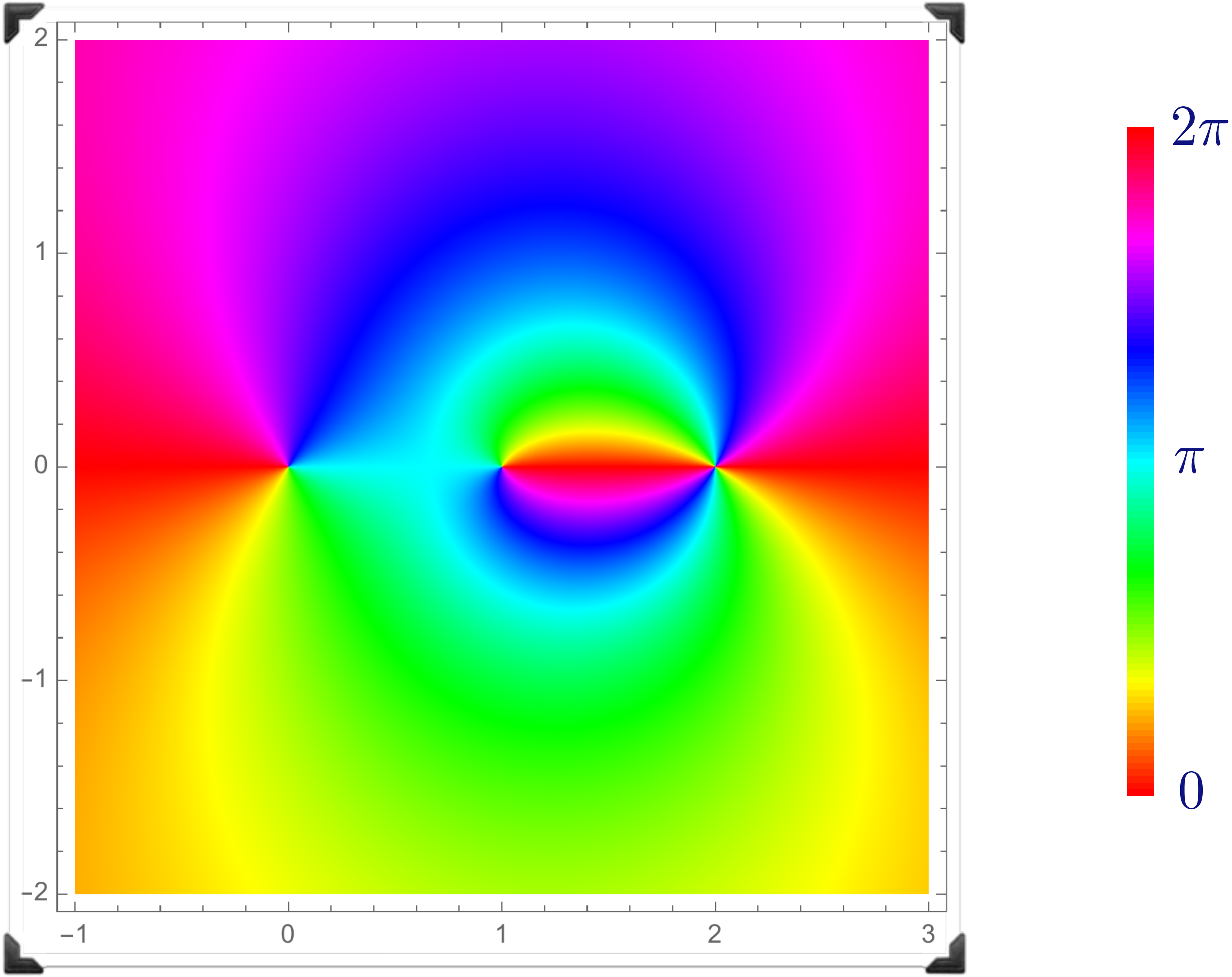
$$w = f(z) = \frac{z^8 - 1}{z - 1}$$



2
1
0
-1
-2
−2
−1
0
1
2
2π
π
0

?

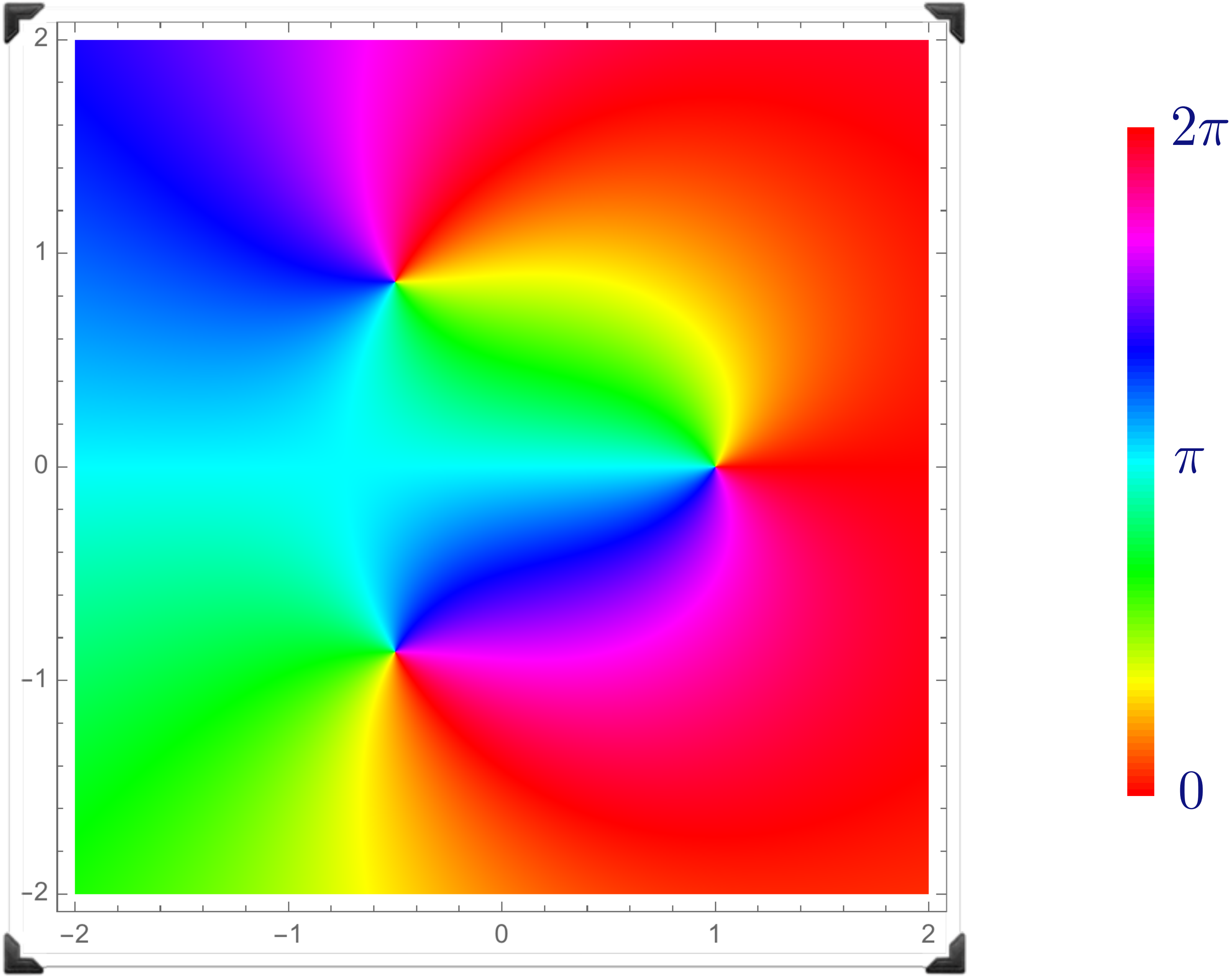
$$w = f(z) = \frac{z^8 - 1}{z^2 - 1}$$



2
1
0
−1
−2
−1
0
1
2
3
2π
π
0

?

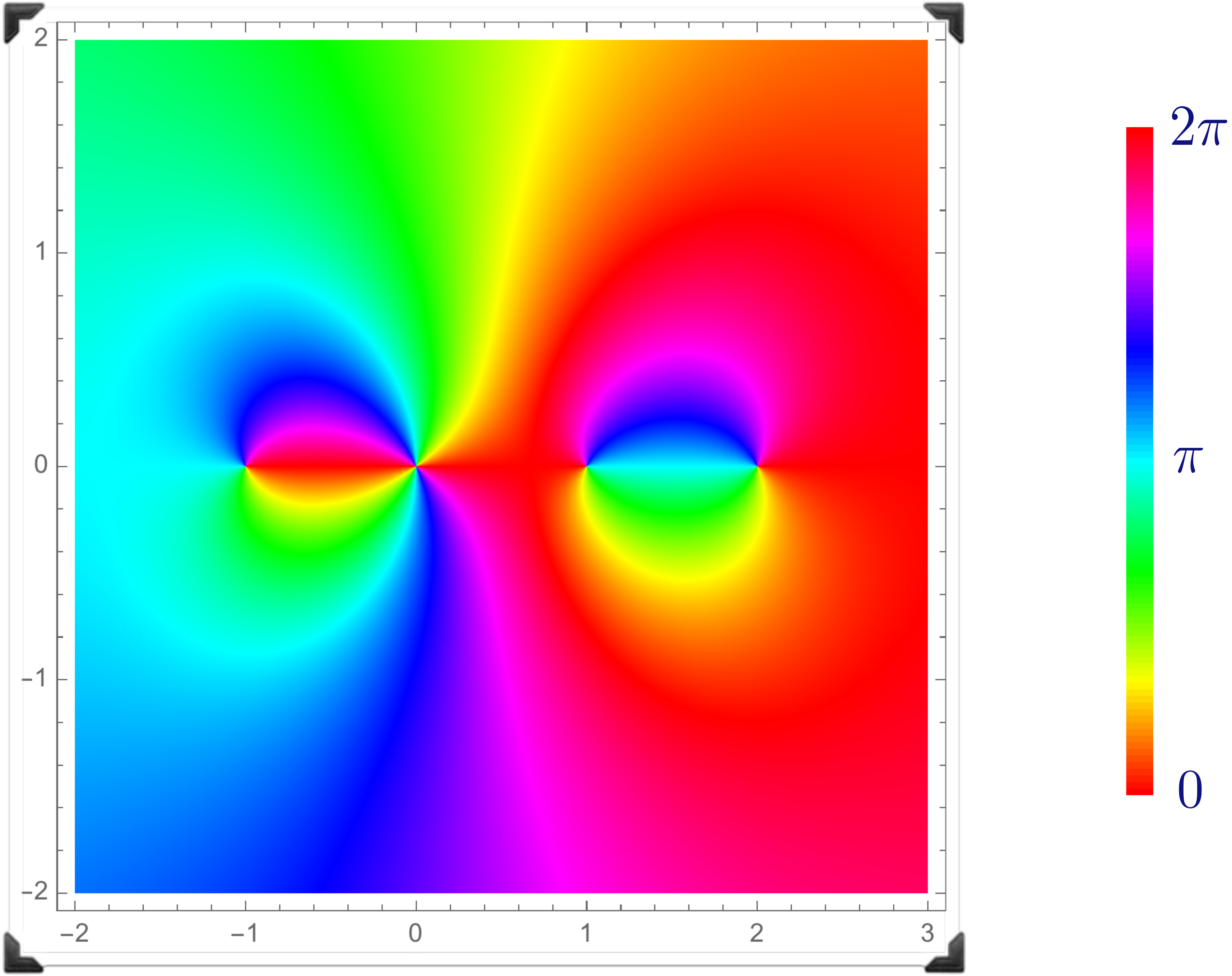
$$w = f(z) = \frac{z(z-1)}{(z-2)^2}$$



2
1
0
−1
−2
−2
−1
0
1
2
2π
π
0

?

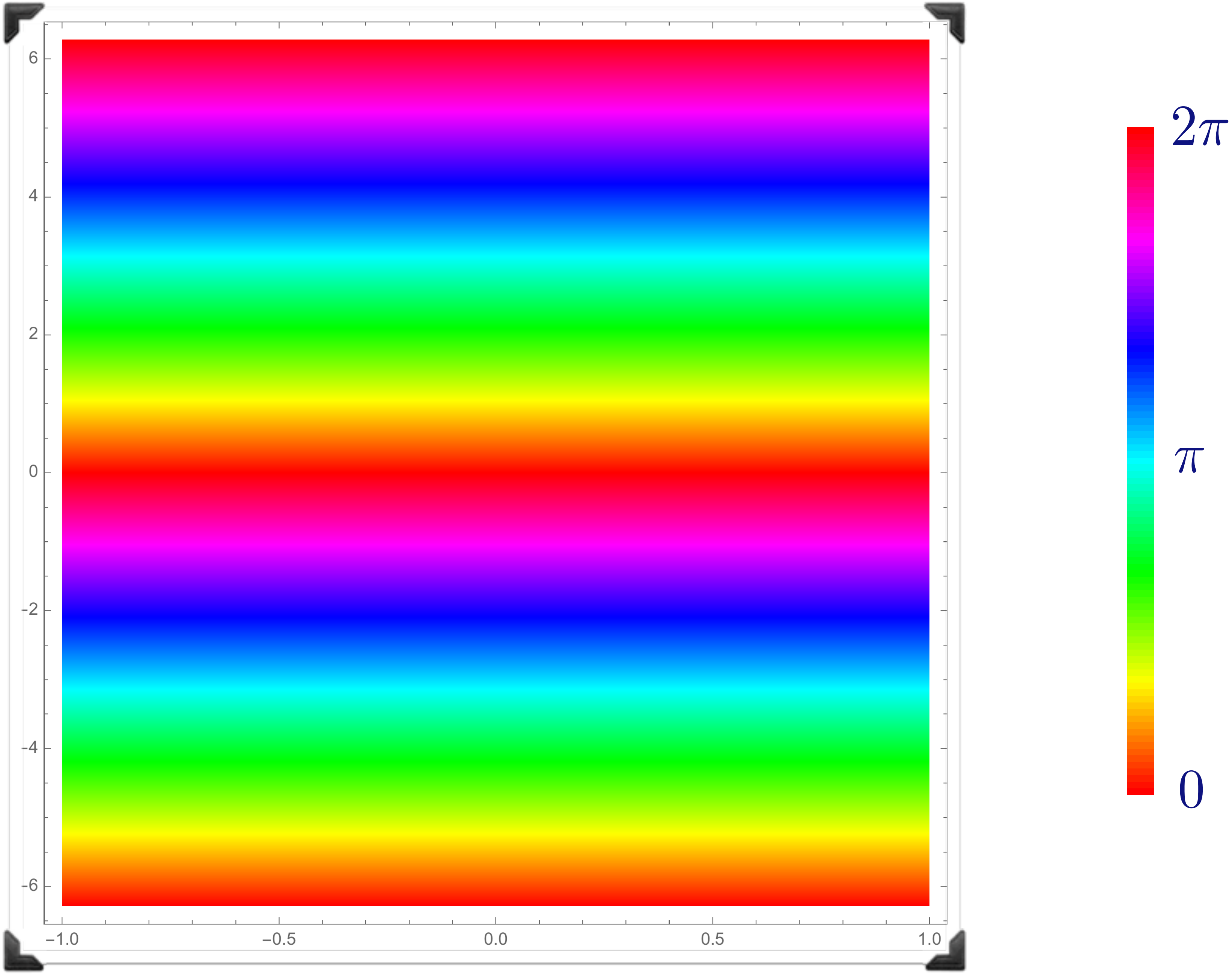
$$w = f(z) = \frac{z - 1}{z^2 + z + 1}$$



2
1
0
−1
−2
−2
−1
0
1
2
3
2π
π
0

?

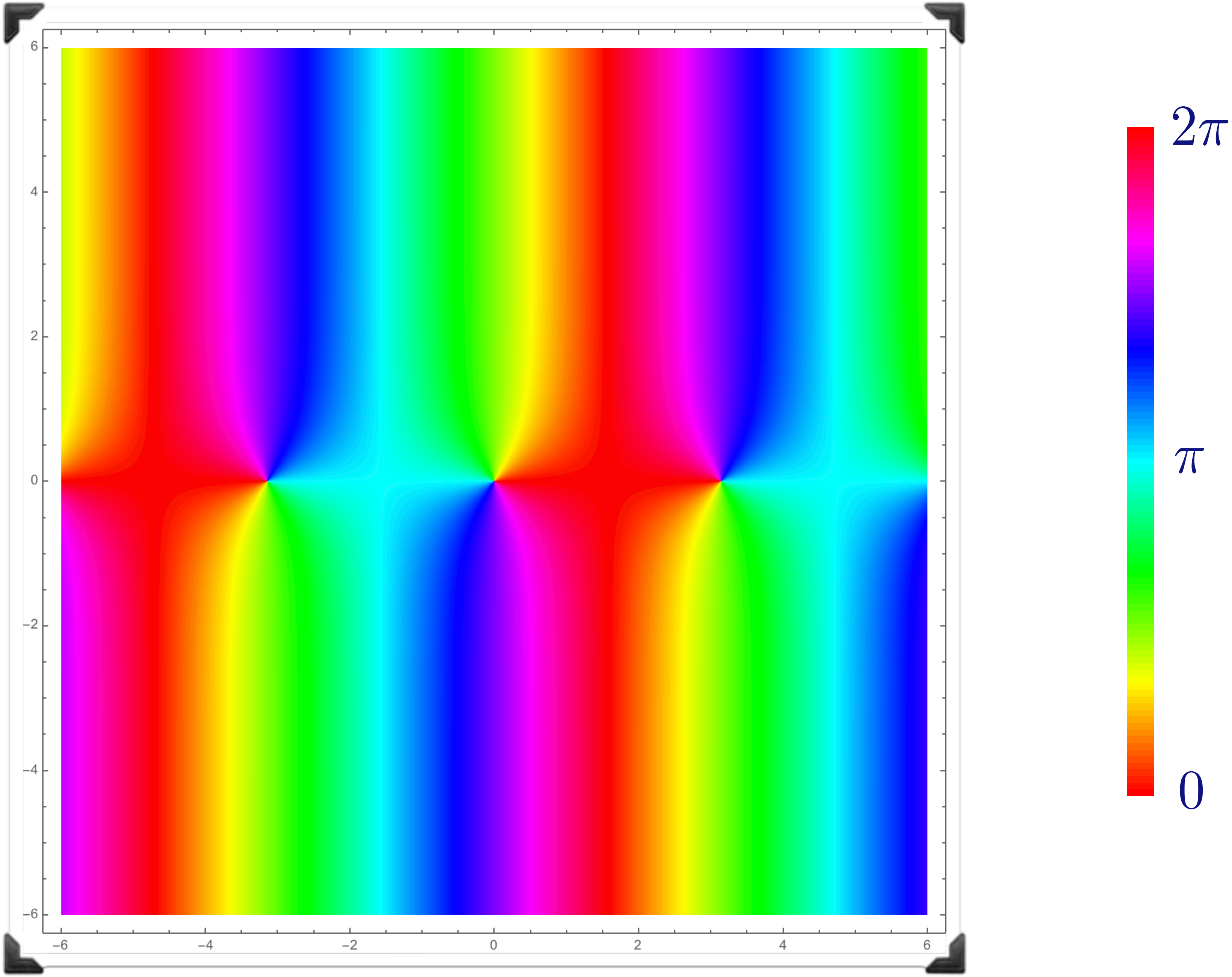
$$w = f(z) = \frac{z^2(z-1)}{(z+1)(z-2)}$$



6
4
2
0
-2
-4
-6
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

?

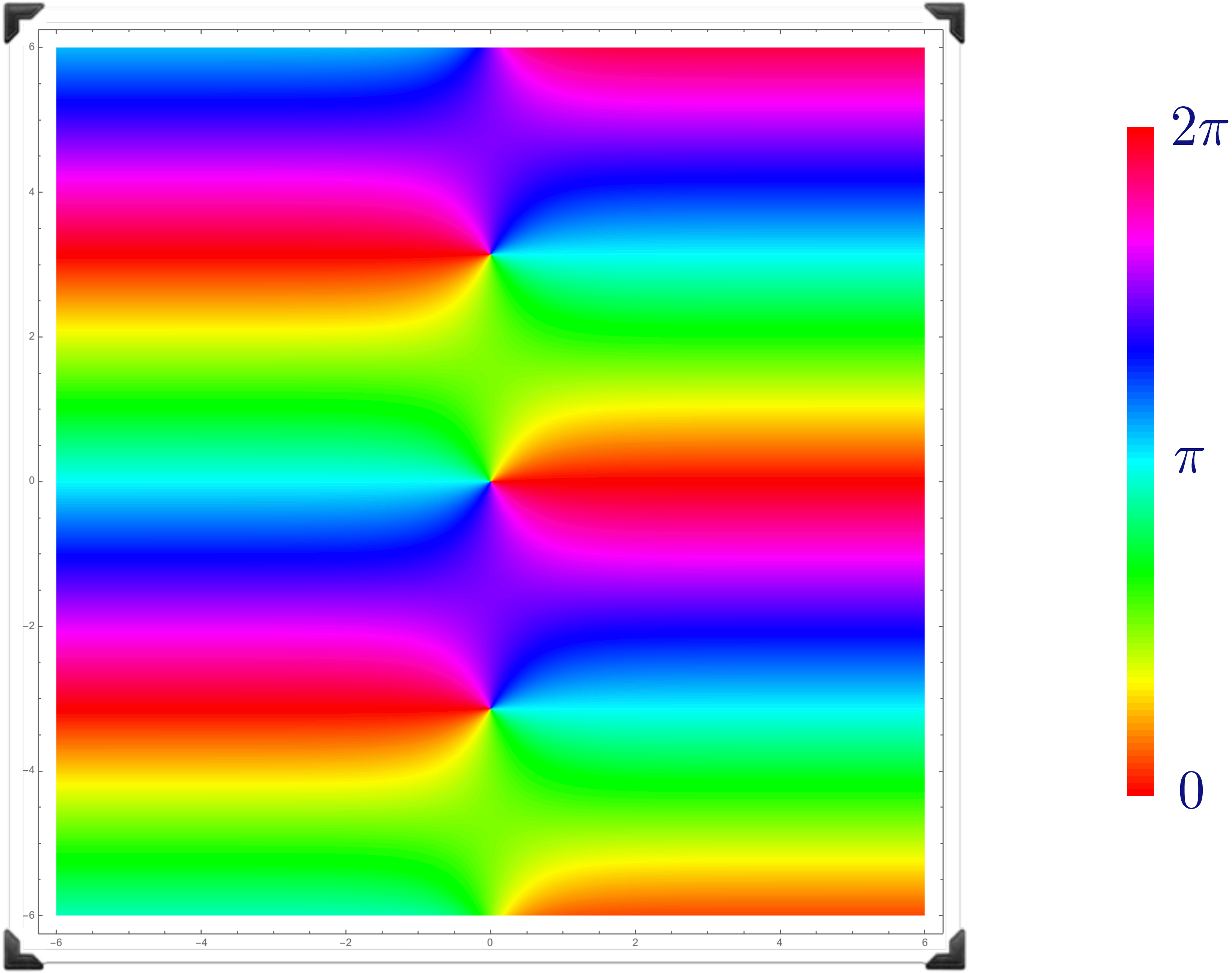
$$w = f(z) = \exp z$$



6
4
2
0
−2
−4
−6
−6
−4
−2
0
2
4
6
2π
π
0

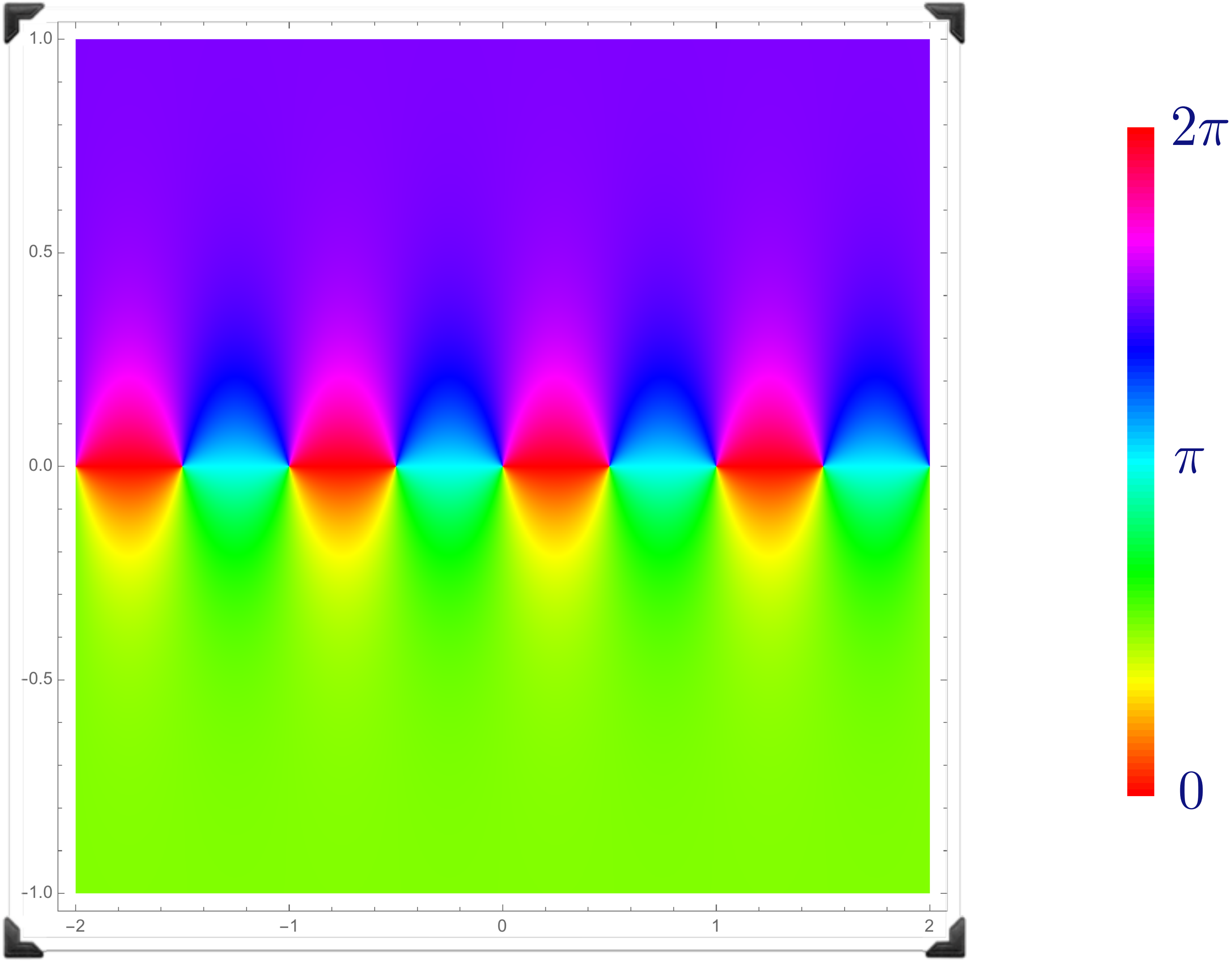
?

$$w = f(z) = \sin z$$


6
4
2
0
−2
−4
−6
−6
−4
−2
0
2
4
6
2π
π
0

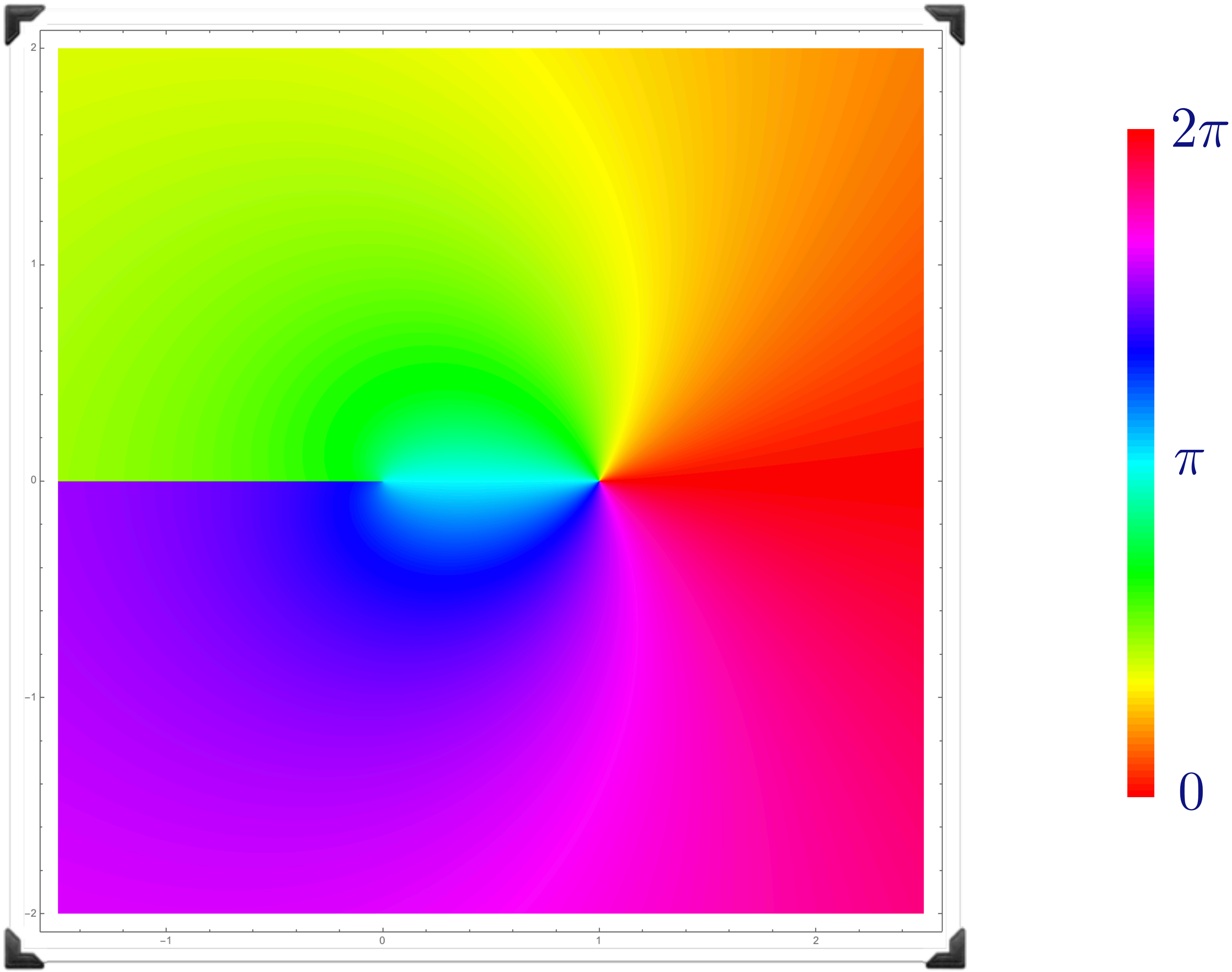
?

$$w = f(z) = \sinh(z)$$


1.0
0.5
0.0
-0.5
-1.0
−2
−1
0
1
2
2π
π
0

?

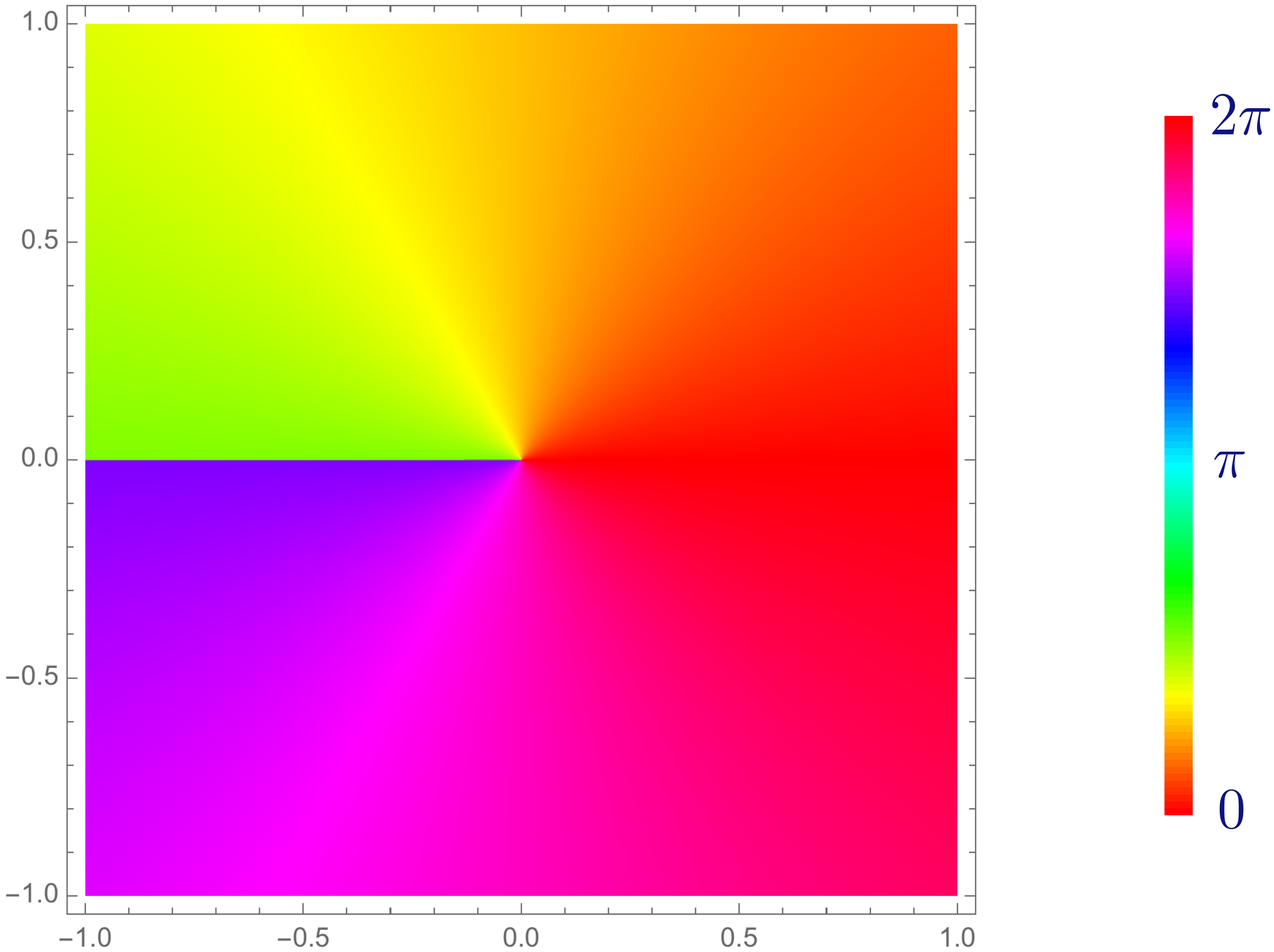
$$w = f(z) = \cot(\pi z)$$



2
1
0
−1
−2
−1
0
1
2
2π
π
0

?

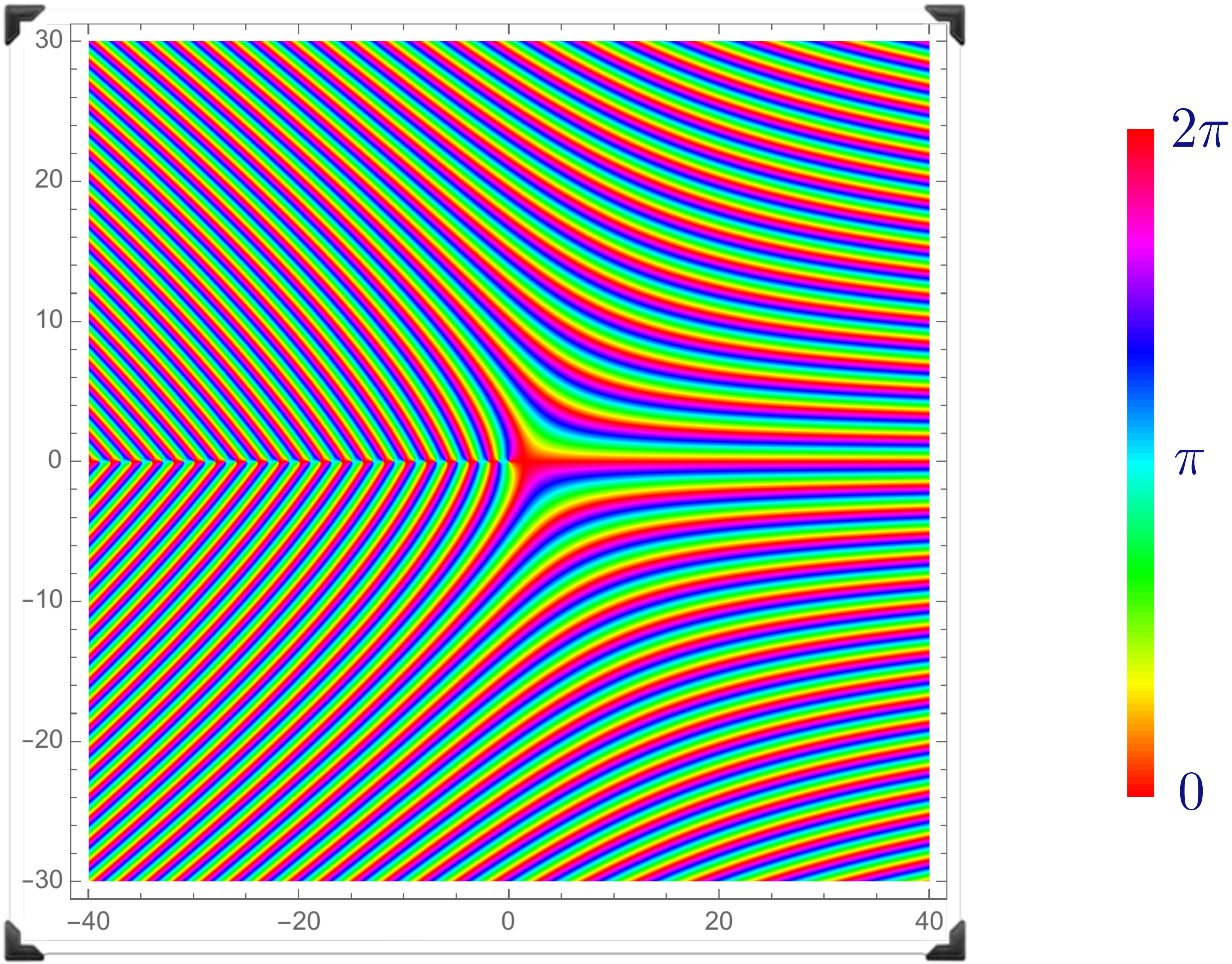
$$w = f(z) = \operatorname{Log} z$$

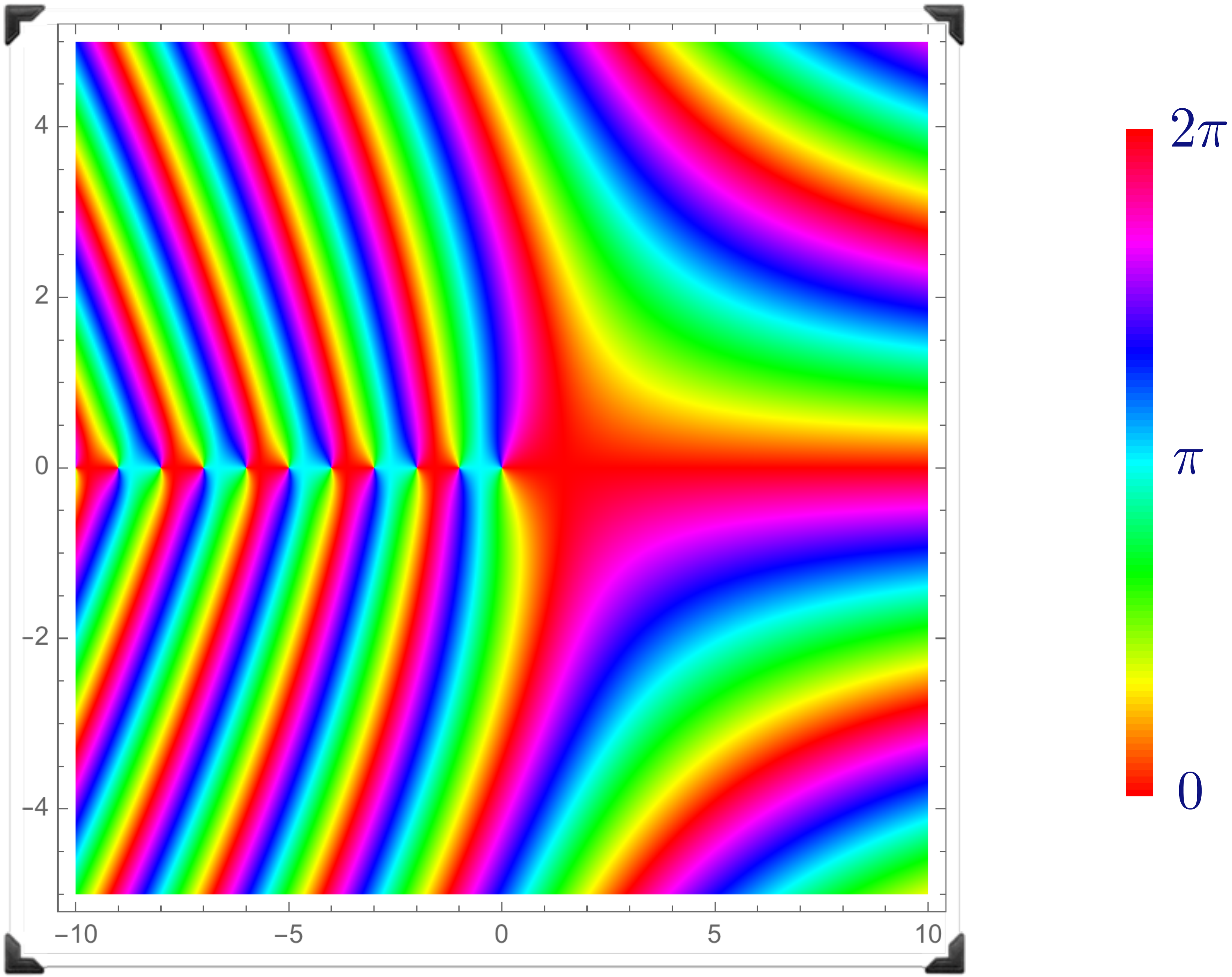


1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0

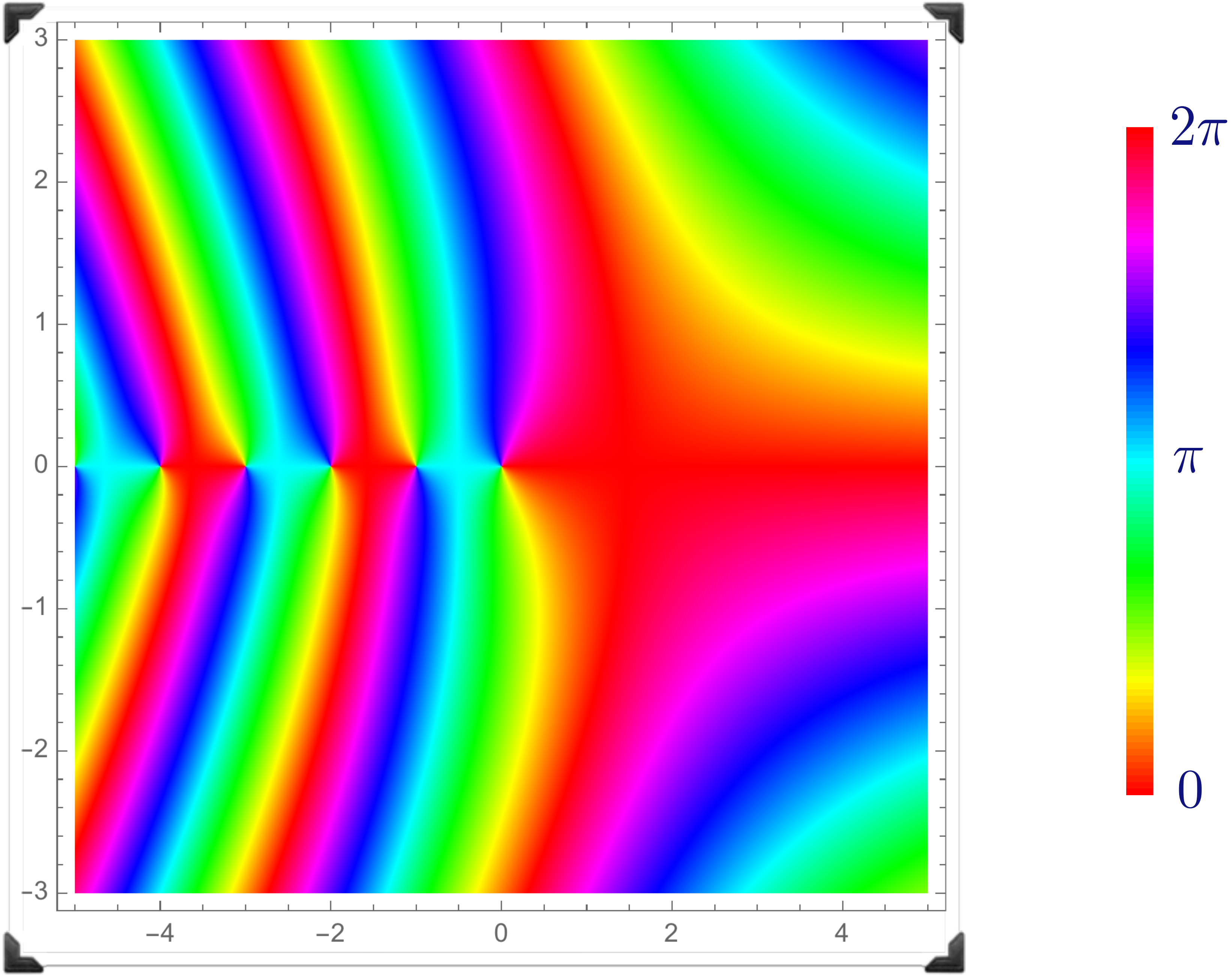
?

$$w = f(z) = \sqrt{z}$$


30
20
10
0
−10
−20
−30
−40
−20
0
20
40
2π
π
0



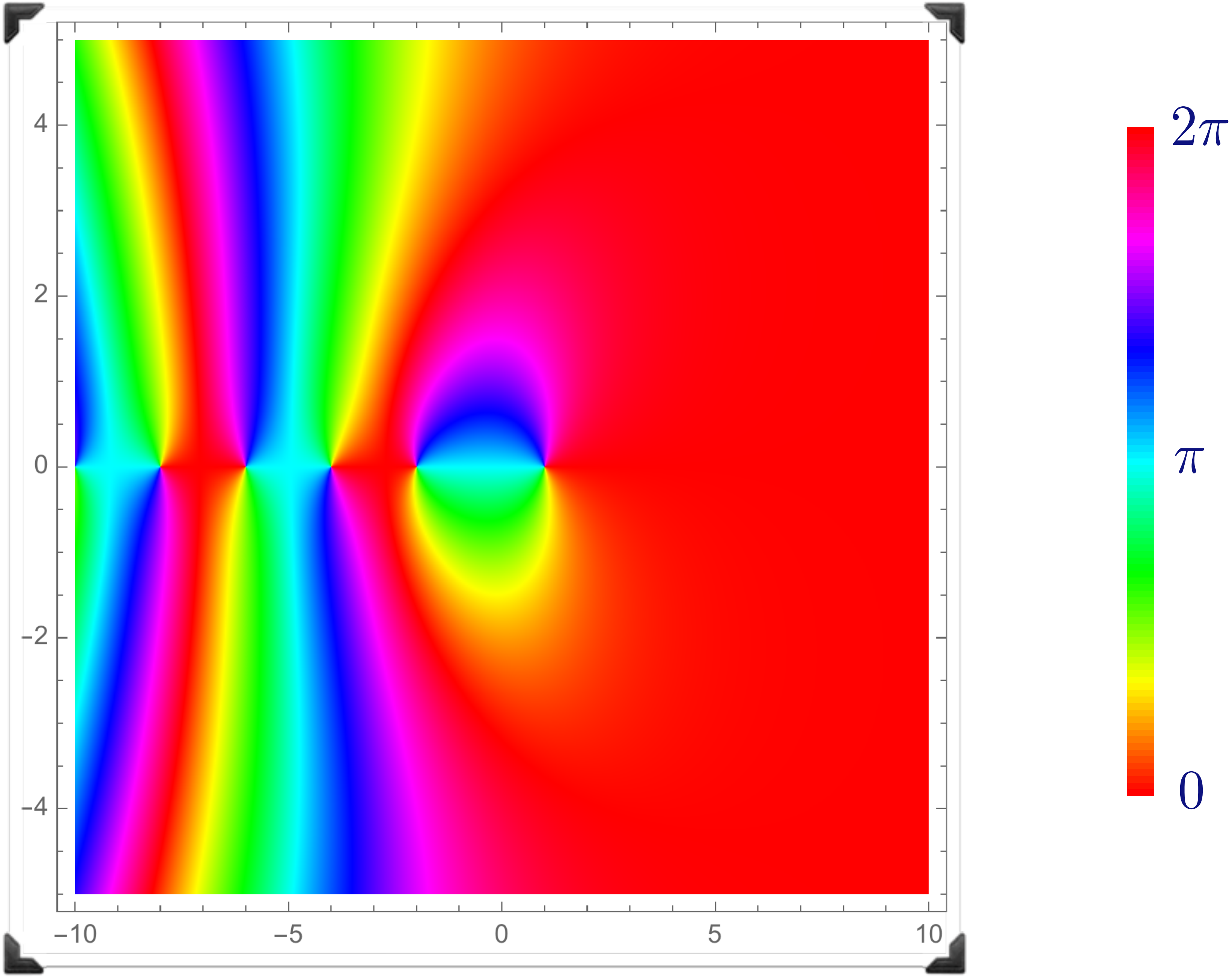
4
2
0
−2
−4
−10
−5
0
5
10
2π
π
0


3
2
1
0
−1
−2
−3
−4
−2
0
2
4
2π
π
0

?

$$
\begin{aligned}
w = f(z) &= \Gamma(z) \\
&= \int_0^\infty t^{z-1}\, e^{-t}\, dt \qquad \text{for } \operatorname{Re} z > 0
\end{aligned}
$$


30
20
10
0
-10
-20
-30
-40
-20
0
20
40
2π
π
0

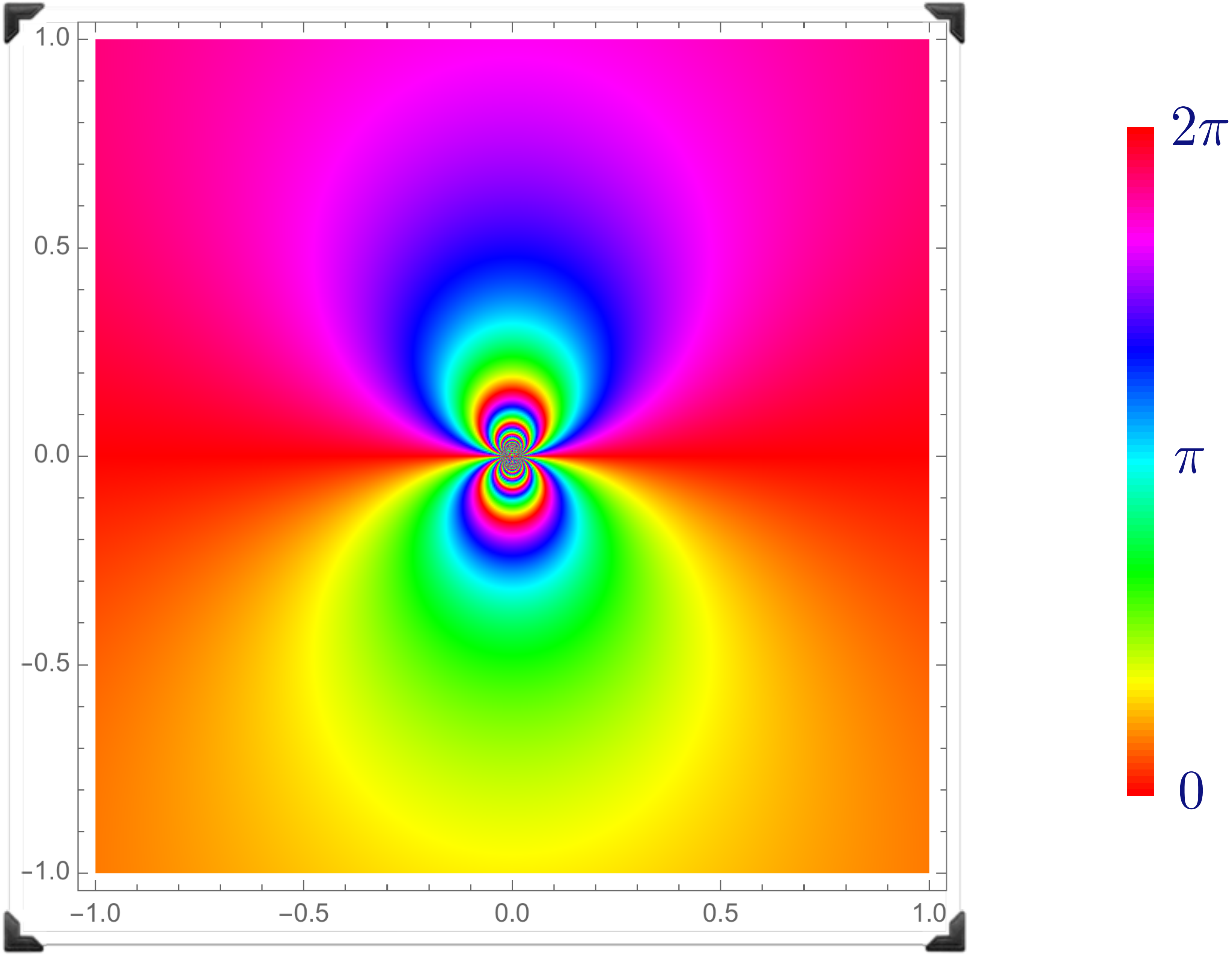


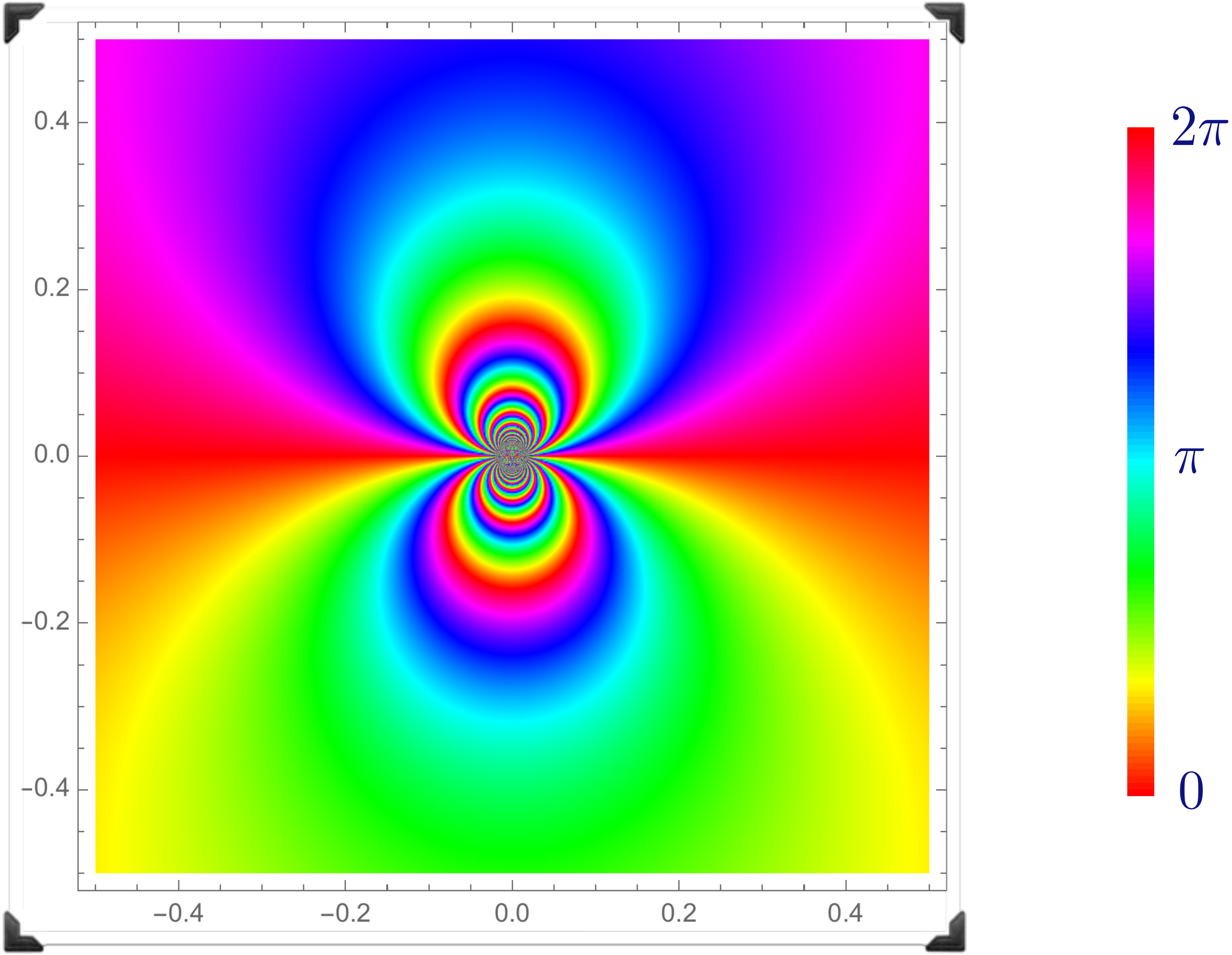
4
2
0
-2
-4
-10
-5
0
5
10
2π
π
0

?

$$w = f(z) = \zeta(z)$$

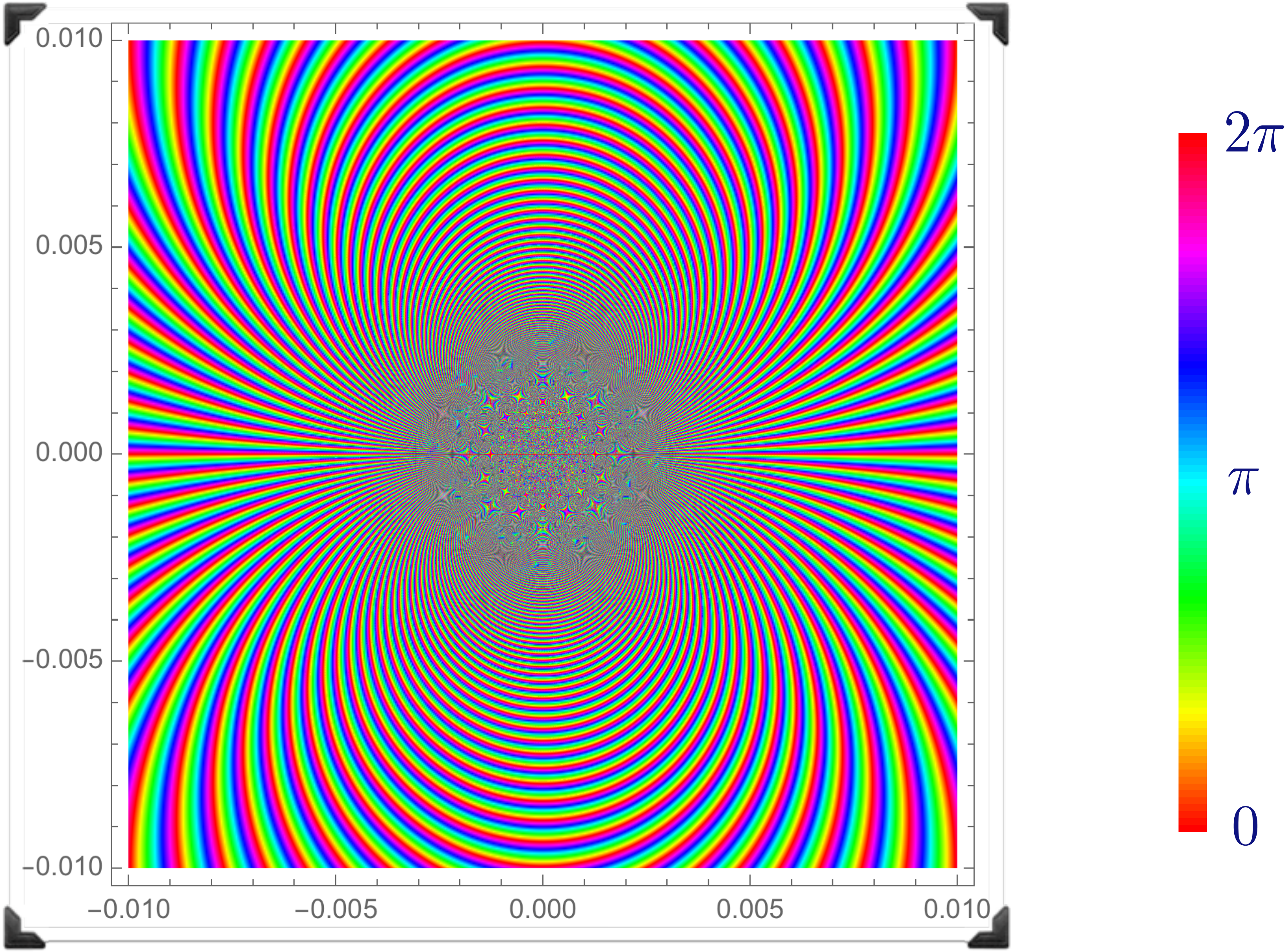
$$= \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad \text{for } \operatorname{Re} z > 1$$

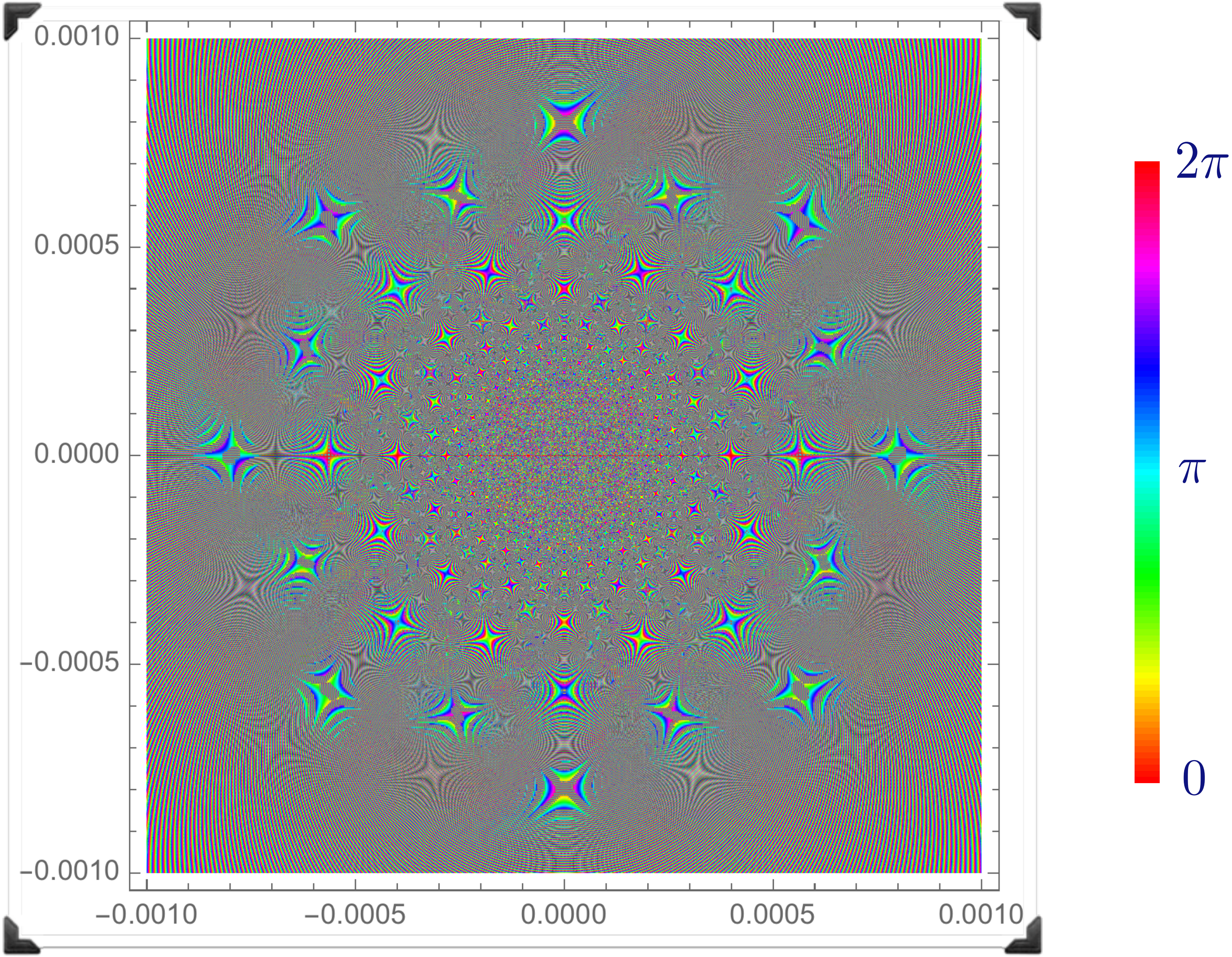

1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0
2π
π
0



0.4
0.2
0.0
−0.2
−0.4
−0.4
−0.2
0.0
0.2
0.4
2π
π
0


0.10
0.05
0.00
−0.05
−0.10
−0.10
−0.05
0.00
0.05
0.10
2π
π
0

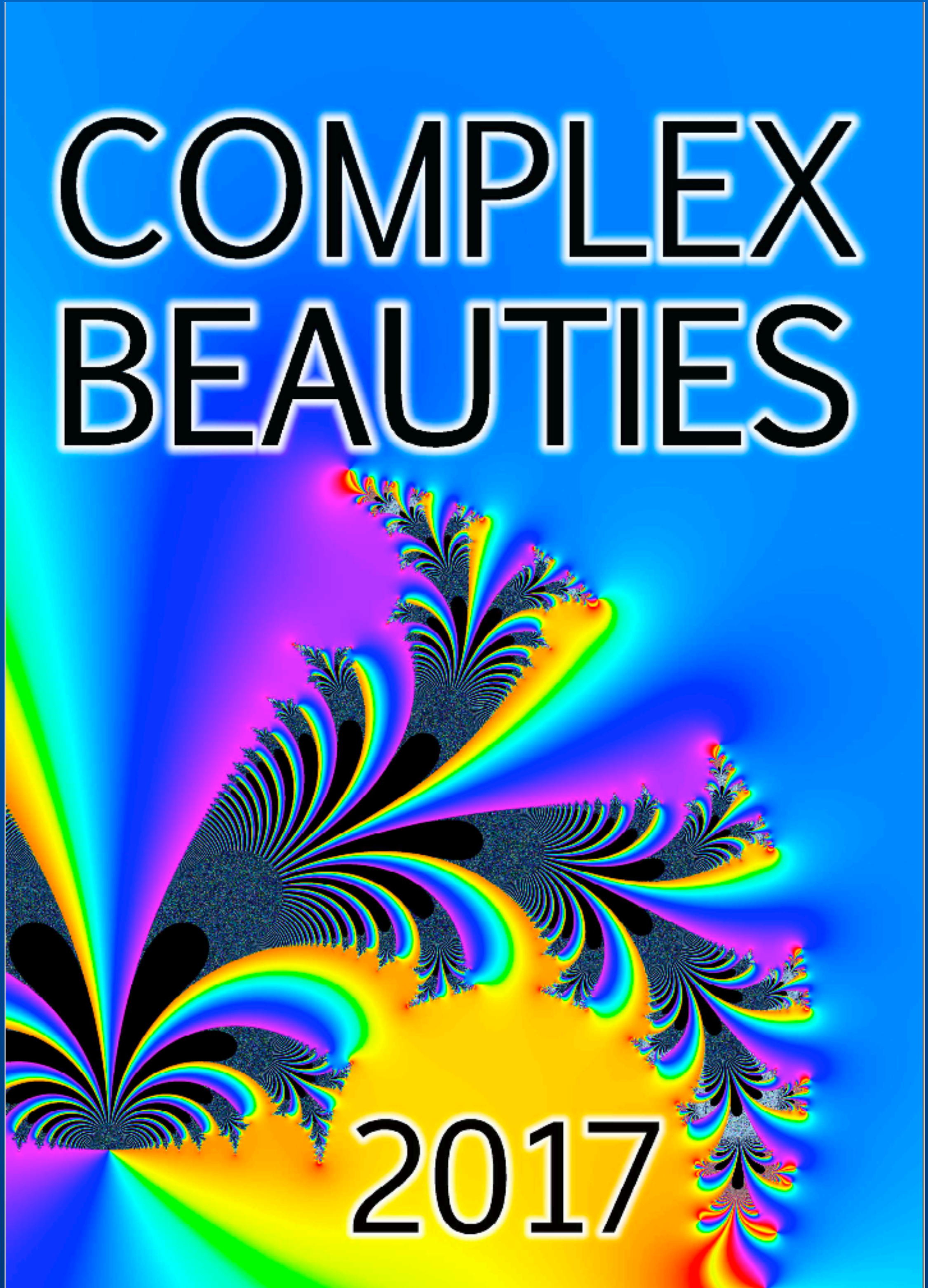

0.010
0.005
0.000
−0.005
−0.010
−0.010
−0.005
0.000
0.005
0.010
2π
π
0



0.0010
0.0005
0.0000
−0.0005
−0.0010
−0.0010
−0.0005
0.0000
0.0005
0.0010
2π
π
0

?

$$w = f(z) = \exp\left(\frac{1}{z}\right)$$


COMPLEX
BEAUTIES
2017

“Complex Beauties” calendar:
http://www.mathe.tu-freiberg.de/files/information/cal17.pdf

Slides made by José Figueroa-O’Farrill & Dennis The
Phase plots here were made in Mathematica.

```
f[z_] := Sqrt[z]

DensityPlot[Rescale[Mod[Arg[f[x + I y]], 2 π], {0, 2 π}], {x, -6, 6}, {y, -6, 6}, PlotPoints → 500,
 ColorFunction → Hue, ColorFunctionScaling → None, Exclusions → None, PerformanceGoal → "Quality"]
```