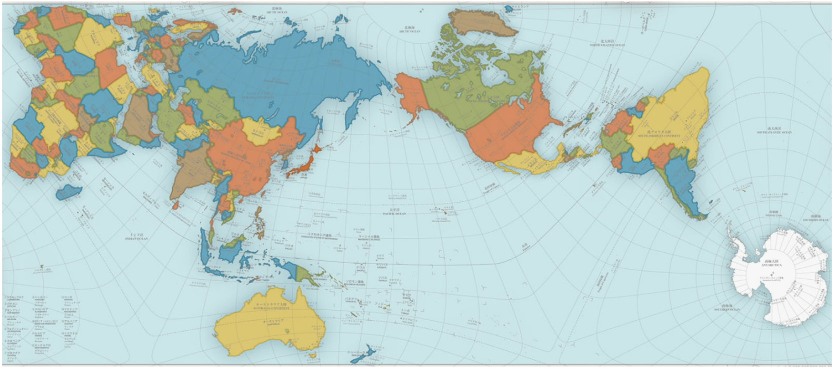
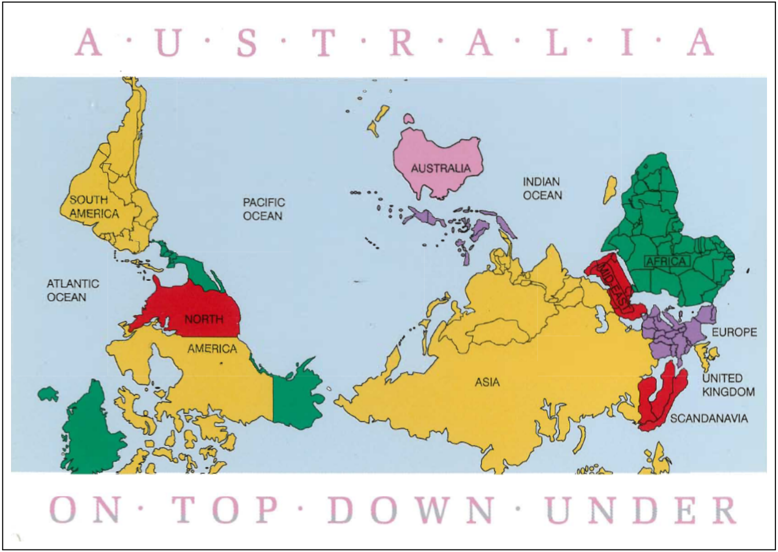
The Mathematics of Maps



(The AuthaGraph world map.)

Dennis The
University of Tromsø
May 2017

A postcard from down under



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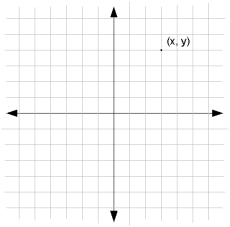
- NB. This is not a geography / history course!

Map projections

Assume: The earth is round. (Actually ellipsoidal, not a sphere.)

$\xrightarrow{?}$



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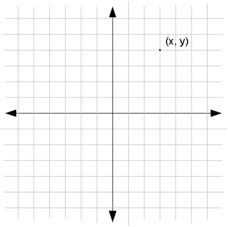

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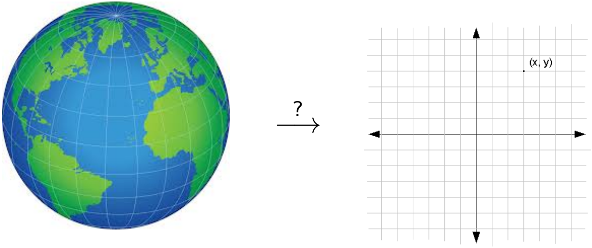

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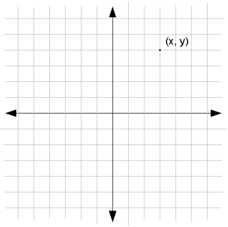

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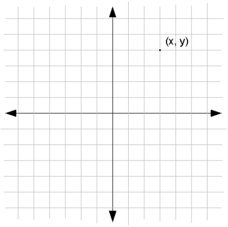

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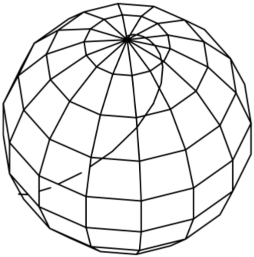
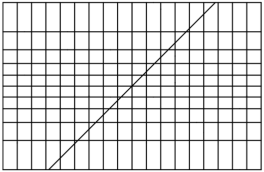


$\mapsto$



parallels of latitude	$\mapsto$	horizontal lines
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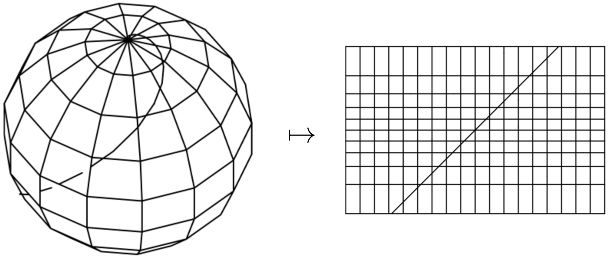

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The last one makes the Mercator map useful for navigation.

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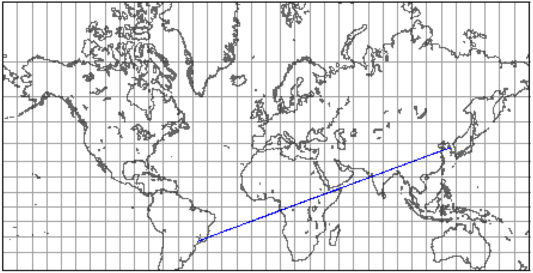

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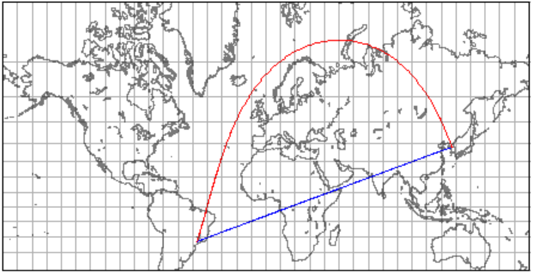

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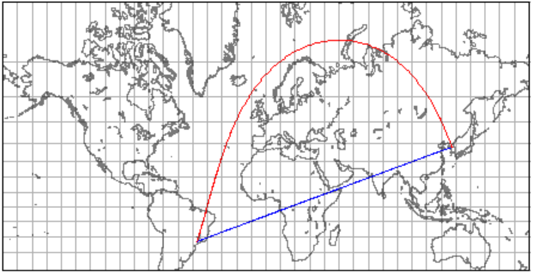
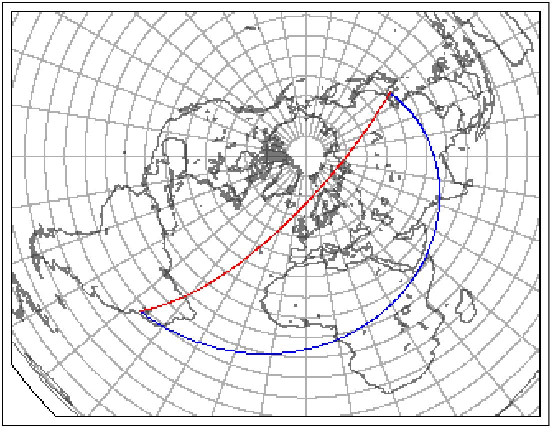


Rhumb line $\rightarrow$ 19450km, Geodesic distance $\rightarrow$ 18290km

Moral: Geodesics are distorted over large scales.

Geodesics - 2

Here are the same paths on an azimuthal equidistant map:



The endpoints are almost antipodes, i.e. opposite on the sphere.

▸ Antipodes

Which has the larger area, and by what factor?

Norway	vs	Thailand	Mexico	vs	New Zealand
Africa	vs	Russia	India	vs	Greenland

Which has the larger area, and by what factor?

Norway	vs	Thailand	Map Fight!	
Mexico	vs	New Zealand	Map Fight!	
Africa	vs	Russia	Map Fight!	
India	vs	Greenland	Map Fight!	

The Gall–Peters (equi-areal) projection

Mercator in the media

"The Mercator projection has fostered European imperialist attitudes for centuries and created an ethnic bias against the third world."

-Organization of cartographers for social equality
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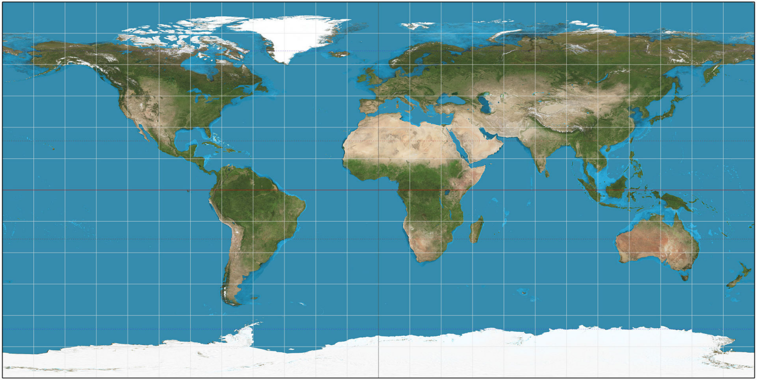
(Mercator is useful for navigation, not for measuring areas!)

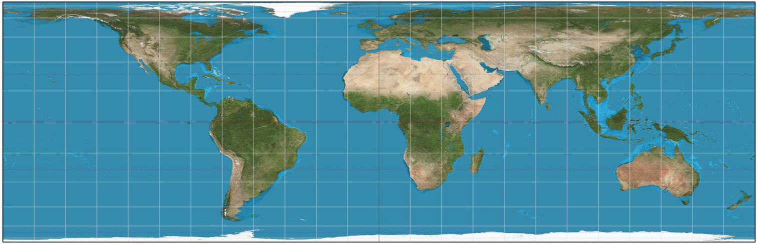
A small gallery of map projections

Mercator map: conformal, rhumb lines go to straight lines.

Equirectangular (plate carrée)



Lambert cylindrical projection



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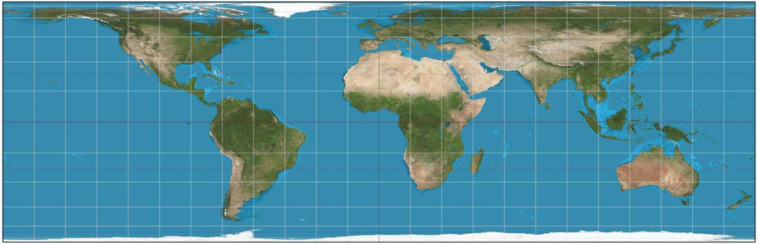

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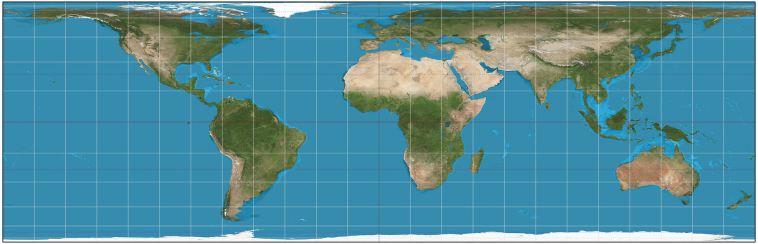

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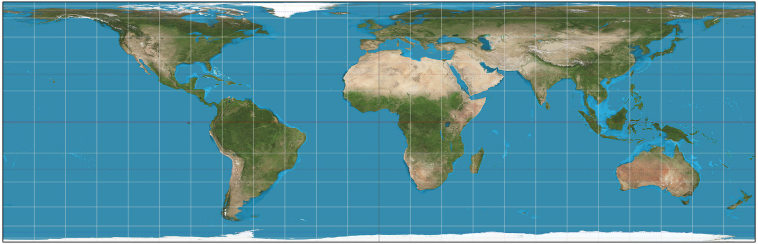

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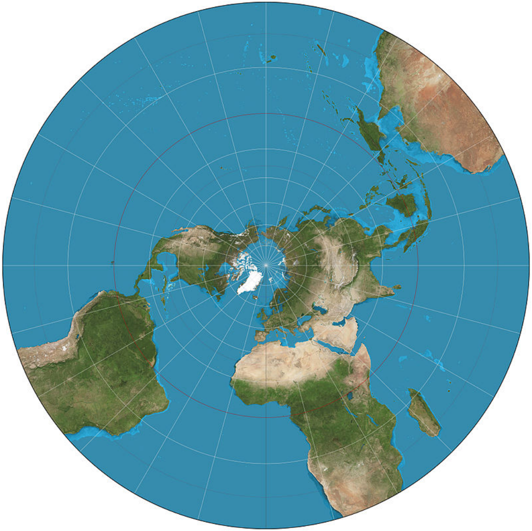
Stereographic projection

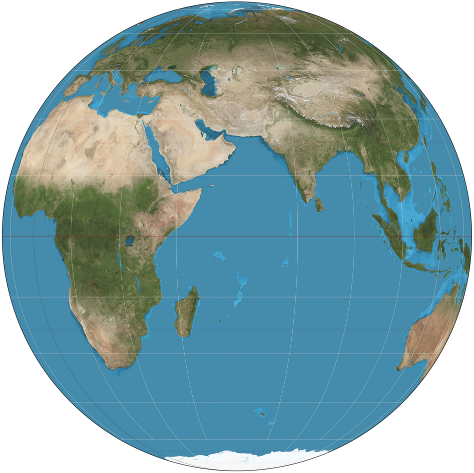


Conformal map, i.e. angles are preserved.

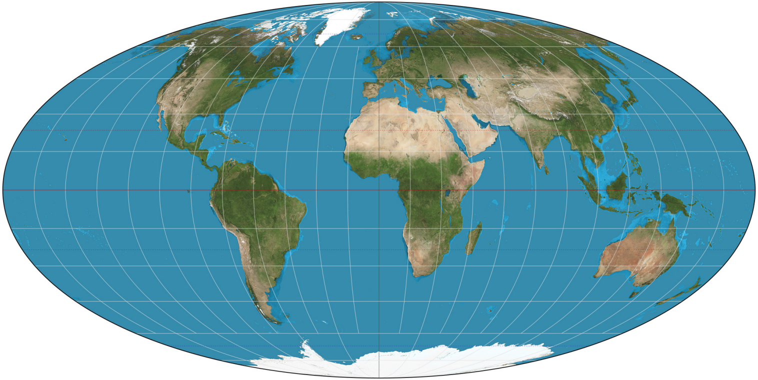
Gnomonic projection

Geodesics are preserved, i.e. (arcs of) great circles go to (segments of) straight lines.

Orthographic projection



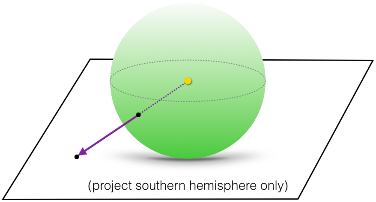
Mollweide projection



Equi-areal

Some "light-based" map projection recipes

Gnomonic



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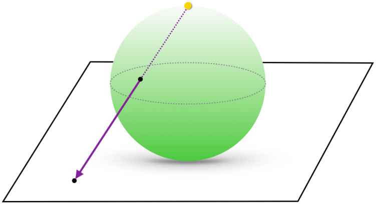

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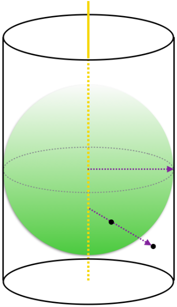

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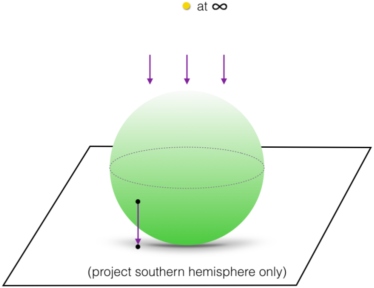
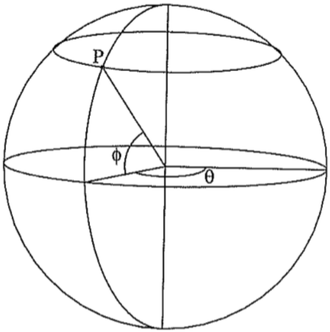


Spherical polar coordinates



$$\begin{cases} x = r\cos\phi\cos\theta \\ y = r\cos\phi\sin\theta \\ z = r\sin\phi \end{cases}$$

Assume unit sphere, so $r = 1$.

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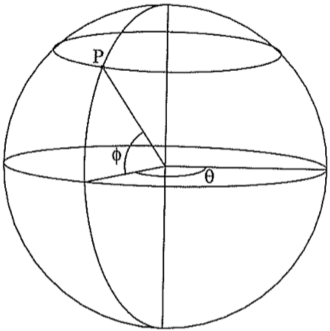

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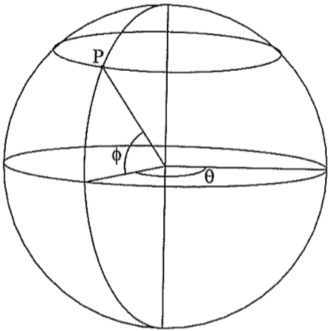

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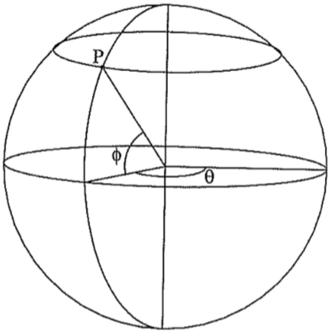

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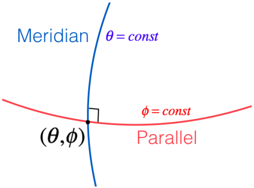
Two basic descriptions for a map Ψ:

Cartesian:	$X = f(\theta,\phi)$,	$Y = g(\theta,\phi)$
Polar:	$R = f(\theta,\phi)$,	$\Theta = g(\theta,\phi)$

Two particular map classes



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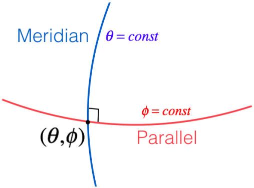
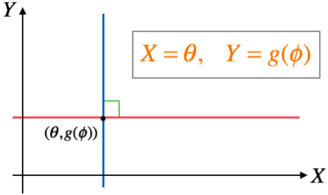
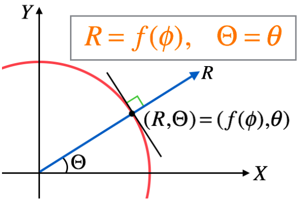


Cylindrical projections	Azimuthal projections
	
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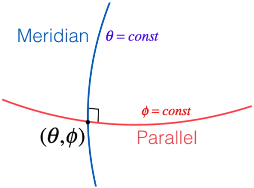

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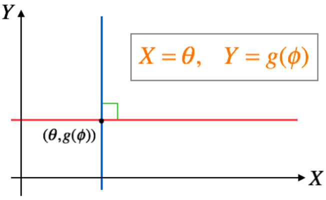

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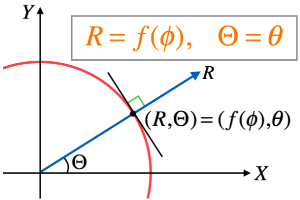

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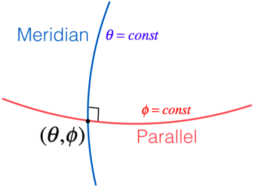

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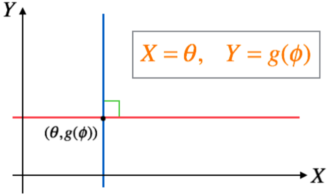	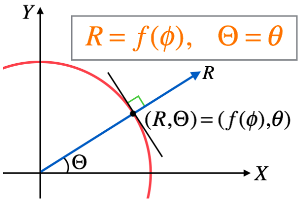
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Images of parallels & meridians are $\perp$. Call these *orthogonal* maps.

Scale factors

Q: How are distances scaled under a map Ψ?

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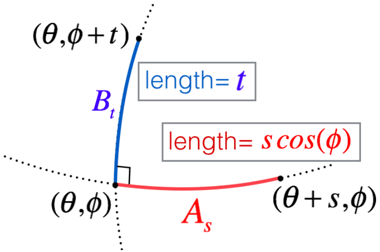

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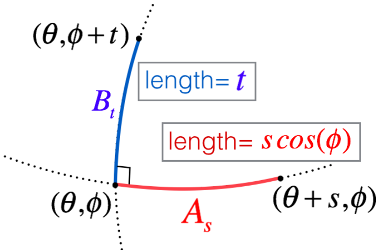

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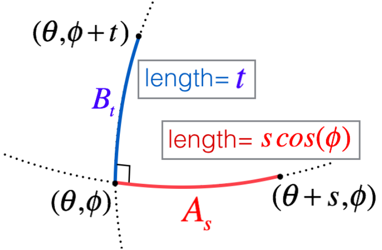


$\overset{\Psi}{\mapsto}$?

IDEA: Ask Q *infinitesimally*.

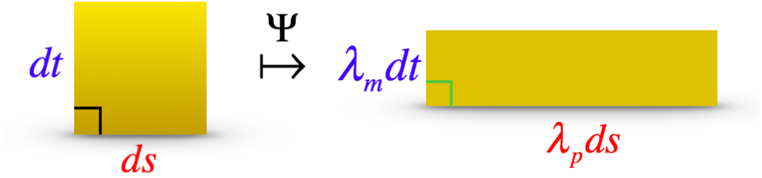
Define scale factors (only depending on (θ, ϕ)):

$$\lambda_p := \lim_{s\to 0} \frac{length(\Psi(A_s))}{length(A_s)}, \qquad \lambda_m := \lim_{t\to 0} \frac{length(\Psi(B_t))}{length(B_t)}.$$

Imposing geometric properties

For orthogonal maps Ψ,

(infinitesimal) rectangles $\mapsto$ (infinitesimal) rectangles



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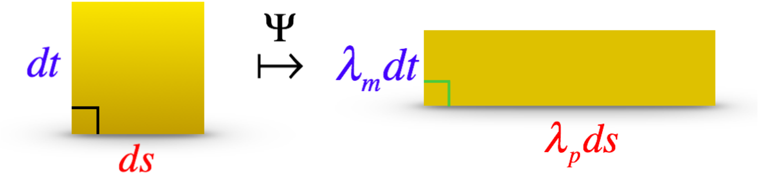

Two natural geometric conditions are:

Equi-areal: Conformal:

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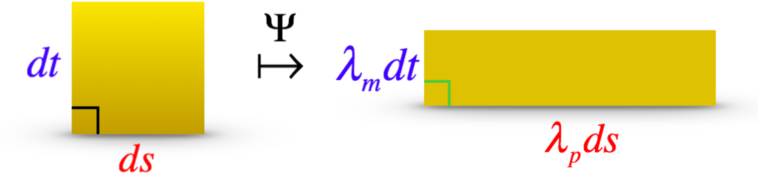

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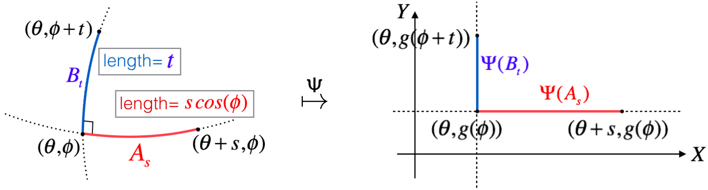
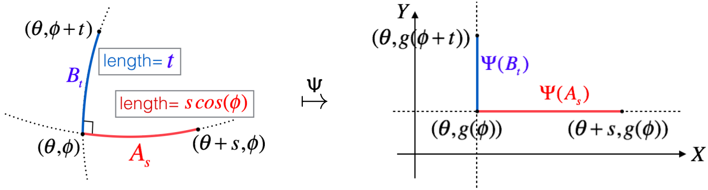
Equi-areal: $\boxed{\lambda_p \lambda_m = 1}$ Conformal: $\boxed{\lambda_p = \lambda_m}$

Cylindrical projections $X = \theta$, $Y = g(\phi)$

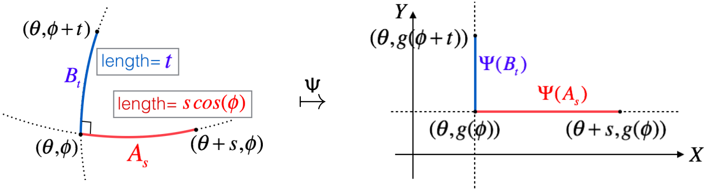


Cylindrical projections $X = \theta$, $Y = g(\phi)$



$$\lambda_p = \lim_{s \to 0} \frac{(\theta + s) - \theta}{s \cos \phi} = \sec \phi, \quad \lambda_m = \lim_{t \to 0} \frac{g(\phi + t) - g(\phi)}{t} = g'(\phi)$$

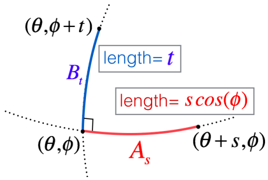
Cylindrical projections $X = \theta$, $Y = g(\phi)$



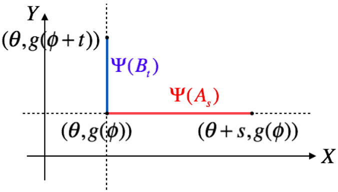
$$\lambda_p = \lim_{s\to 0} \frac{(\theta+s)-\theta}{s\cos\phi} = \sec\phi, \quad \lambda_m = \lim_{t\to 0} \frac{g(\phi+t)-g(\phi)}{t} = g'(\phi)$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$g'(\phi) = \sec(\phi)$	
Equi-areal $\lambda_m \lambda_p = 1$	$g'(\phi) = \cos(\phi)$	

Cylindrical projections $X = \theta$, $Y = g(\phi)$



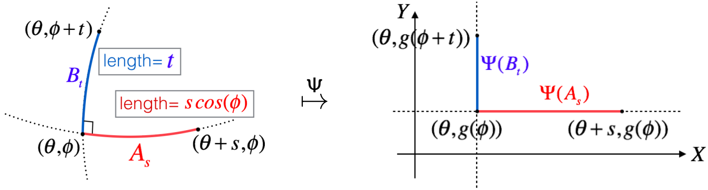
$\overset{\Psi}{\mapsto}$



$$\lambda_p = \lim_{s\to 0} \frac{(\theta+s)-\theta}{s\cos\phi} = \sec\phi, \qquad \lambda_m = \lim_{t\to 0} \frac{g(\phi+t)-g(\phi)}{t} = g'(\phi)$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$g'(\phi) = \sec(\phi)$	Mercator
Equi-areal $\lambda_m\lambda_p = 1$	$g'(\phi) = \cos(\phi)$	

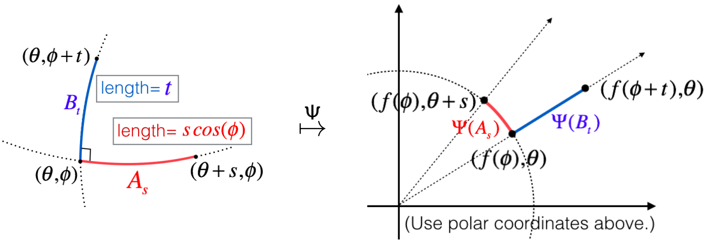
Cylindrical projections $X = \theta$, $Y = g(\phi)$



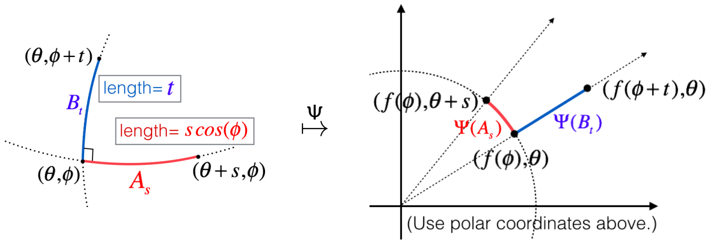
$$\lambda_p = \lim_{s \to 0} \frac{(\theta + s) - \theta}{s \cos \phi} = \sec \phi, \quad \lambda_m = \lim_{t \to 0} \frac{g(\phi + t) - g(\phi)}{t} = g'(\phi)$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$g'(\phi) = \sec(\phi)$	Mercator
Equi-areal $\lambda_m \lambda_p = 1$	$g'(\phi) = \cos(\phi)$	Lambert cylindrical

Azimuthal projection $R = f(\phi)$, $\Theta = \theta$

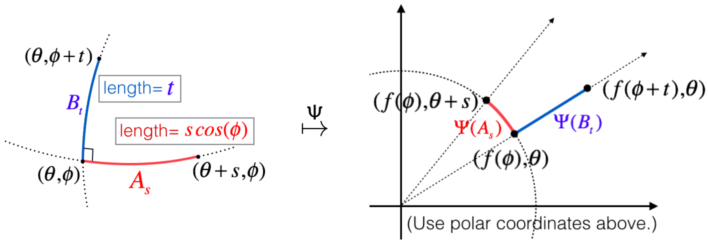


Azimuthal projection $R = f(\phi)$, $\Theta = \theta$



$$\lambda_p = \lim_{s \to 0} \frac{f(\phi)s}{s \cos \phi} = f(\phi) \sec(\phi), \quad \lambda_m = \lim_{t \to 0} \frac{f(\phi + t) - f(\phi)}{t} = f'(\phi).$$

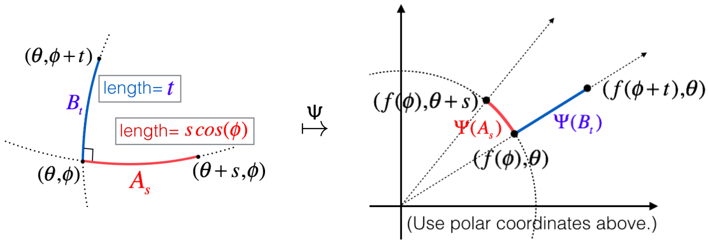
Azimuthal projection $R = f(\phi)$, $\Theta = \theta$



$$\lambda_p = \lim_{s\to 0} \frac{f(\phi)s}{s\cos\phi} = f(\phi)\sec(\phi), \quad \lambda_m = \lim_{t\to 0} \frac{f(\phi+t)-f(\phi)}{t} = f'(\phi).$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$f'(\phi) = f(\phi)\sec(\phi)$	
Equi-areal $\lambda_m\lambda_p = 1$	$f'(\phi)f(\phi) = \cos(\phi)$	

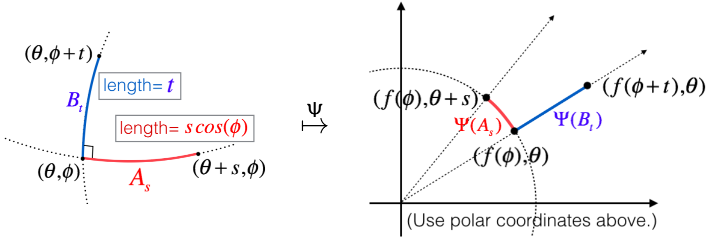
Azimuthal projection $R = f(\phi)$, $\Theta = \theta$



$$\lambda_p = \lim_{s \to 0} \frac{f(\phi)s}{s\cos\phi} = f(\phi)\sec(\phi), \qquad \lambda_m = \lim_{t \to 0} \frac{f(\phi+t) - f(\phi)}{t} = f'(\phi).$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$f'(\phi) = f(\phi)\sec(\phi)$	Stereographic
Equi-areal $\lambda_m \lambda_p = 1$	$f'(\phi)f(\phi) = \cos(\phi)$	

Azimuthal projection $R = f(\phi)$, $\Theta = \theta$



$$\lambda_p = \lim_{s \to 0} \frac{f(\phi)s}{s \cos \phi} = f(\phi)\sec(\phi), \quad \lambda_m = \lim_{t \to 0} \frac{f(\phi + t) - f(\phi)}{t} = f'(\phi).$$

Property	Differential equation	Resulting map
Conformal $\lambda_m = \lambda_p$	$f'(\phi) = f(\phi)\sec(\phi)$	Stereographic
Equi-areal $\lambda_m \lambda_p = 1$	$f'(\phi)f(\phi) = \cos(\phi)$	Lambert azimuthal

MoMS will continue...

MoMS (Mathematics of Maps Seminar) next week:

- Ideal maps do not exist! (aka. "All maps must lie.") Why?
- Geodesics and the gnomonic projection.