

The Mathematics of Maps – Lecture 2

WHAT YOUR FAVORITE
MAP PROJECTION
SAYS ABOUT YOU

MERCATOR



YOU'RE NOT REALLY INTO MAPS.

ROBINSON



YOU HAVE A COMFORTABLE PAIR OF RUNNING SHOES THAT YOU WEAR EVERYWHERE. YOU LIKE COFFEE AND ENJOY THE BEATLES. YOU THINK THE ROBINSON IS THE BEST-LOOKING PROJECTION, HANDS DOWN.

WINKEL-TRIPLEL



NATIONAL GEOGRAPHIC ADOPTED THE WINKEL-TRIPLEL IN 1998, BUT YOU'VE BEEN A WIT FAN SINCE LONG BEFORE 'NAT GEO' SHOWED UP. YOU'RE WORRIED ITS GETTING PLAYED OUT AND ARE THINKING OF SWITCHING TO THE KAYRASKY. YOU ONCE LEFT A PARTY IN DISGUST WHEN A GUEST SHOWED UP WEARING SHOES WITH IDEAS. YOUR FAVORITE MUSICAL GENRE IS "POST".

VAN DER GRINTEN



YOU'RE NOT A COMPLICATED PERSON. YOU LOVE THE MERCATOR PROJECTION; YOU JUST WISH IT WEREN'T SQUARE. THE EARTH'S NOT A SQUARE, IT'S A CIRCLE. YOU LIKE CIRCLES. TEDDY'S GONNA BE A GOOD DAD!

DYMAXION



YOU LIKE ISAPIC PASHON, X-MIL, AND SHOES WITH TEES. YOU THINK THE SEAGUY GOT A BAD RAP. YOU OWN 3D GOOGLES, WHICH YOU USE TO VIEW ROTATING MODELS OF BETTER 3D GOOGLES. YOU TYPE IN DUURK.

GOODE HOMOLOSINE



THEY SAY MAPPING THE EARTH ON A 2D SURFACE IS LIKE FLATTENING AN ORANGE PEEL, WHICH SEEMS EASY ENOUGH TO YOU. YOU LIKE EASY SOLUTIONS. YOU THINK WE WOULDN'T HAVE SO MANY PROBLEMS IF WE'D JUST ELECT ADOPTABLE PEOPLE TO CONGRESS INSTEAD OF POLITICIANS. YOU THINK AIRLINES SHOULD JUST BUY FOOD FROM THE RESTAURANTS NEAR THE GATES AND SERVE THAT ON BOARD. YOU CHANGE YOUR OHS ON, BUT SECRETLY WONDER IF YOU REALLY MEZZ TO.

HOB0-DYER



YOU WANT TO AVOID CULTURAL IMPERIALISM, BUT YOU'VE HEARD BAD THINGS ABOUT GAIL-PETERS. YOU'RE CONFLICT-AVERSE AND BUY ORGANIC. YOU USE A RECENTLY-INVENTED SET OF GENDER-NEUTRAL PRONOUNS AND THINK THAT WHAT THE WORLD NEEDS IS A REVOLUTION IN CONSCIOUSNESS.

A GLOBE!



YES, YOU'RE VERY CLEVER.

PEIRCE QUINCUNAL



YOU THINK THAT WHEN WE LOOK AT A MAP, WHAT WE REALLY SEE IS OURSELVES. AFTER YOU FIRST SAW 'INCEPTION', YOU SAT SLUMBER IN THE THEATER FOR SIX HOURS. IT TAKES YOU OUT TO REALIZE THAT EVERYONE AROUND YOU HAS A SKELETON INSIDE THEM. YOU HAVE REALLY LOOKED AT YOUR HANDS.

PLATE CARREEE
(RECTANGULAR)



YOU THINK THIS ONE IS FINE. YOU LIKE HOW X AND Y MAP TO LATITUDE AND LONGITUDE. THE OTHER PROJECTIONS OVERCOMPLICATE THINGS. YOU WANT ME TO STOP ASKING ABOUT MAPS SO YOU CAN ENJOY DINNER.

WATERMAN BUTTERRY



REALLY? YOU KNOW THE WATERMAN? HAVE YOU SEEN THE 1909 CHALL MAP ITS BASED... YOU HAVE A FRAMED REPRODUCTION AT HOME?! WHOA... LISTEN, FORGET THESE QUESTIONS. ARE YOU DOING ANYTHING TONIGHT?

GALL-PETERS



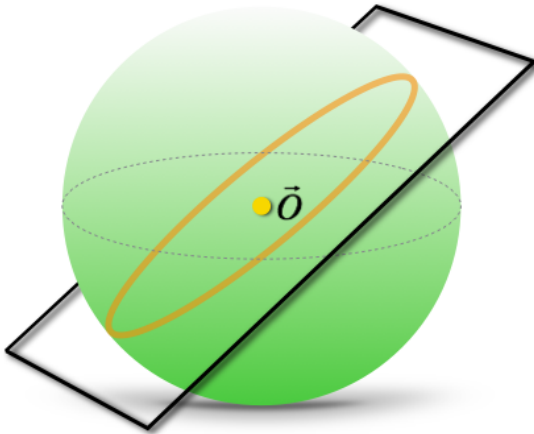
I HATE YOU.

(Source: [xkcd comics](#))

All maps must lie

Great circles

A **great circle** is the intersection of the sphere with a plane passing through the sphere's center. These are **geodesics** on the sphere.



Ideal maps

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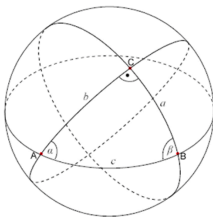
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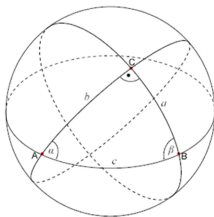
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Spherical triangles



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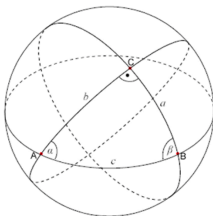


Theorem (Area of a spherical triangle)

Let ΔABC be a *spherical* triangle (i.e. edges are great circle arcs) on a sphere of radius r . If Σ is its internal angle sum, then

$$0 < \text{Area}(\Delta ABC) = (\Sigma - \pi)r^2.$$

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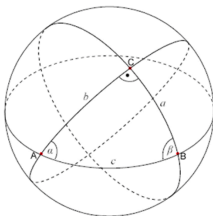
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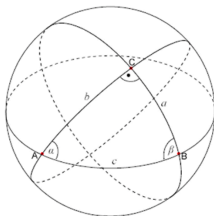
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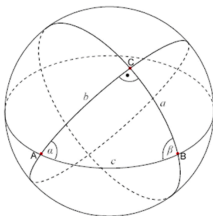
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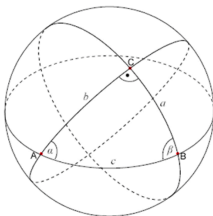
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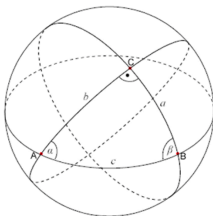
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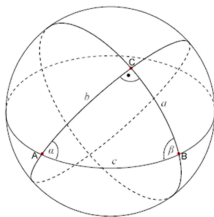
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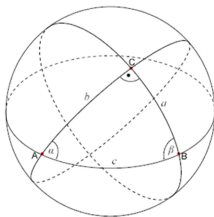
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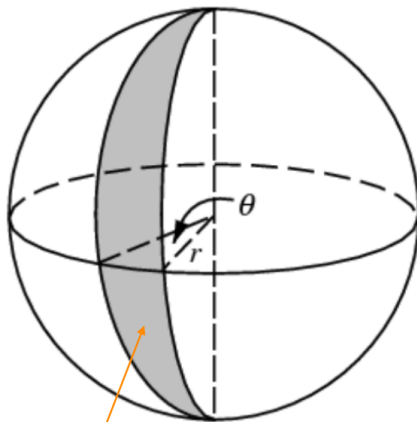
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Q: Where does the above AST theorem come from?

Lunes

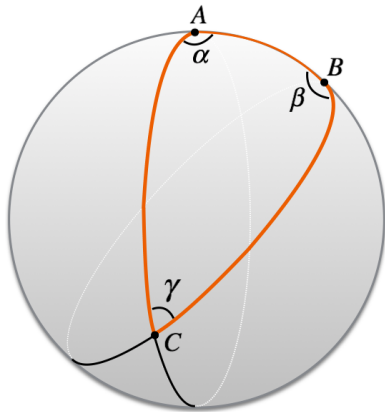
Lune = portion between two intersecting great circles and the antipodal points where they cross.



$$\text{area} = \frac{\theta}{2\pi} (4\pi r^2) = 2\theta r^2$$

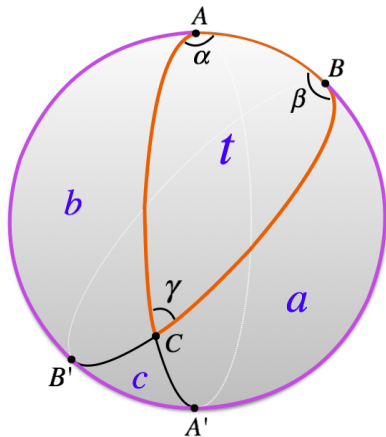
Understanding the AST theorem

Given a spherical $\triangle ABC$, rotate so that arc AB looks as below.



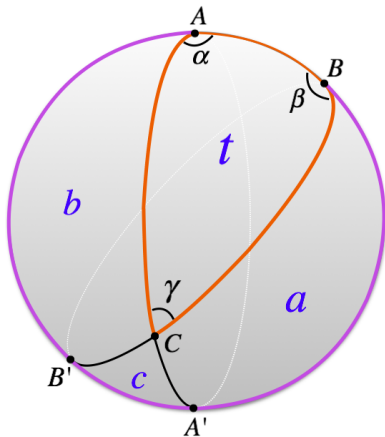
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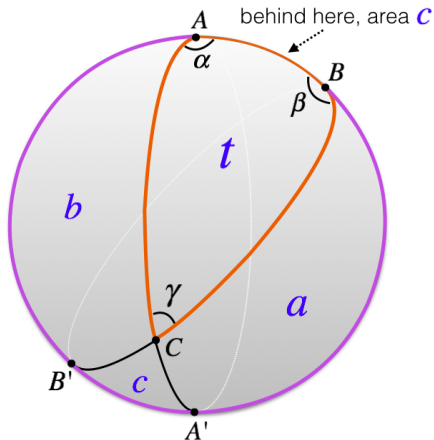
$$a + t = 2\alpha r^2,$$

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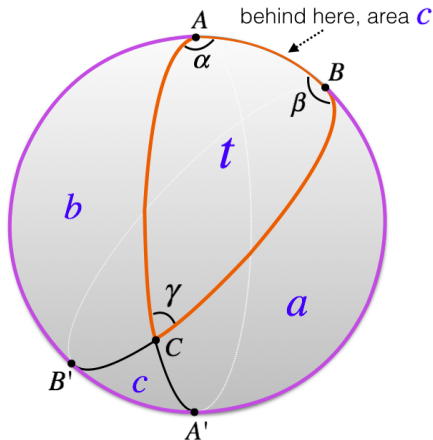
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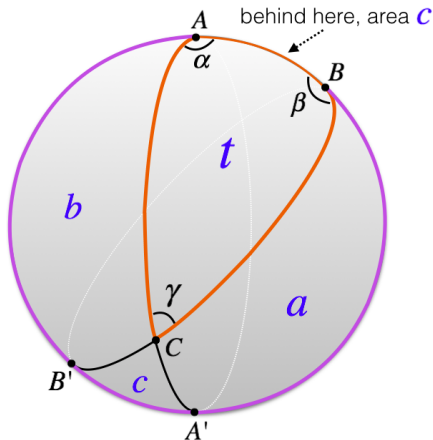
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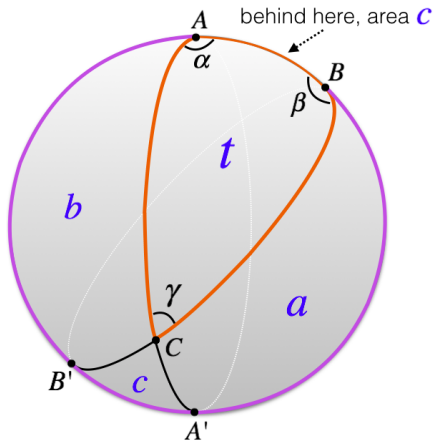
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Subtract & $\div 2$:

$$t = (\Sigma - \pi)r^2. \therefore \text{AST}\checkmark$$

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Recall (D): Distances are rescaled by the same constant factor λ .

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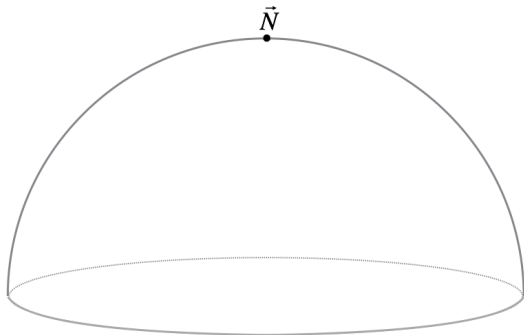
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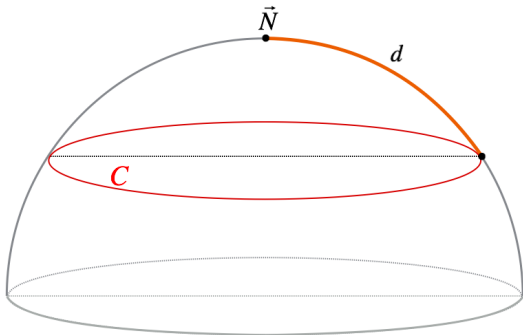
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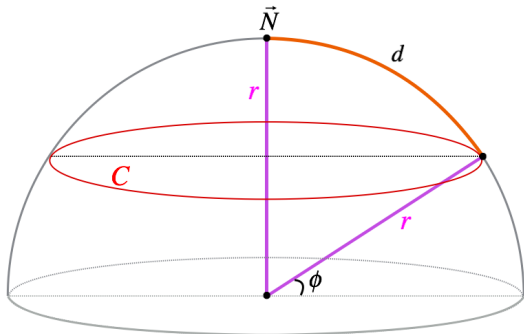
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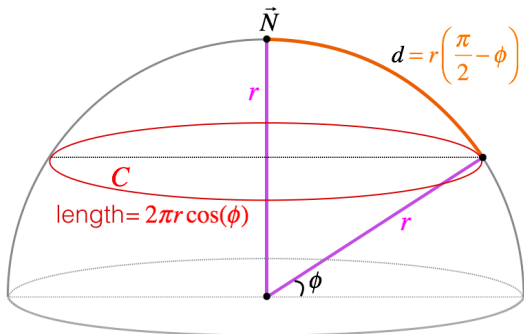
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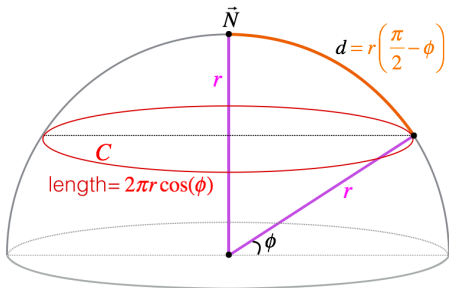
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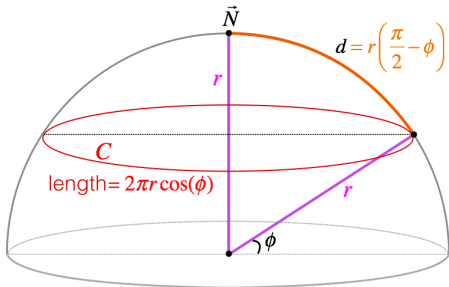


Euler's proof - 2



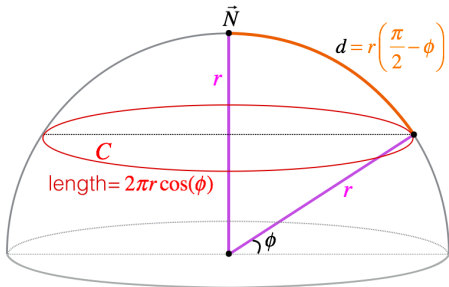
$$length(C) = 2\pi r \cos \phi$$

Euler's proof - 2



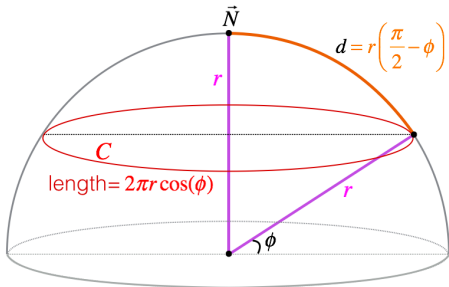
$$\text{length}(C) = 2\pi r \cos \phi = 2\pi r \sin \left(\frac{\pi}{2} - \phi \right)$$

Euler's proof - 2



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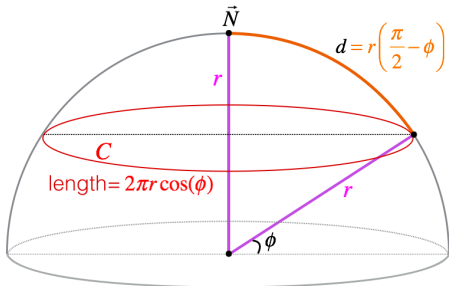
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$$\text{length}(C) = 2\pi r \cos \phi = 2\pi r \sin \left(\frac{\pi}{2} - \phi \right) = 2\pi r \sin \left(\frac{d}{r} \right) < 2\pi d,$$

using the fact that $\sin(x) < x$ for any $x > 0$. (Exercise.)

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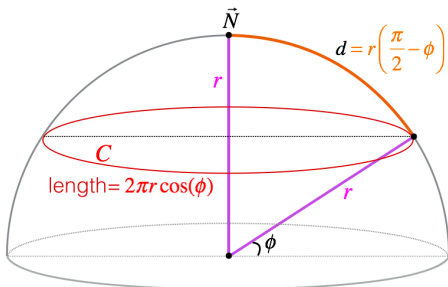


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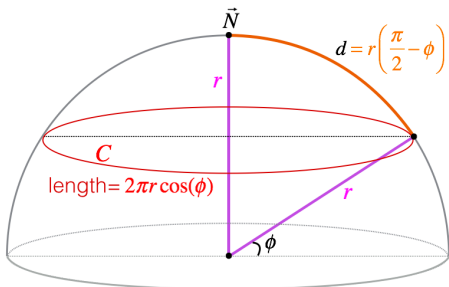


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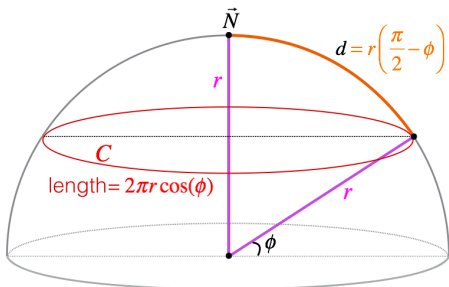


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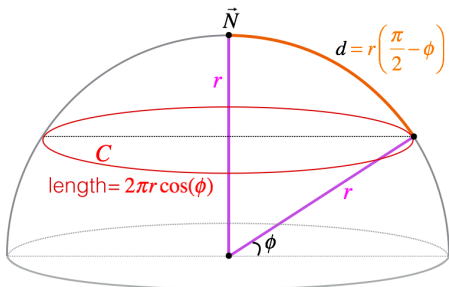
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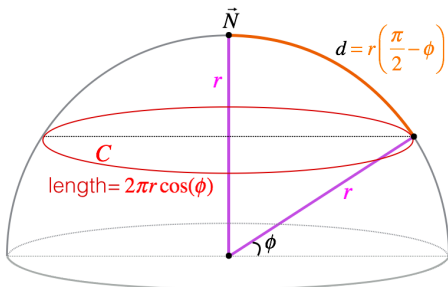
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Contradiction!

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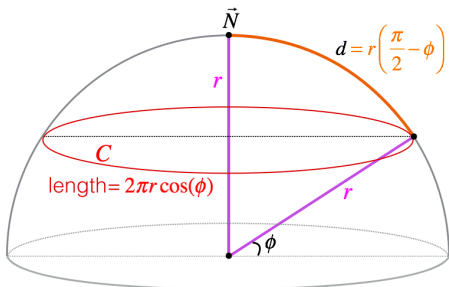
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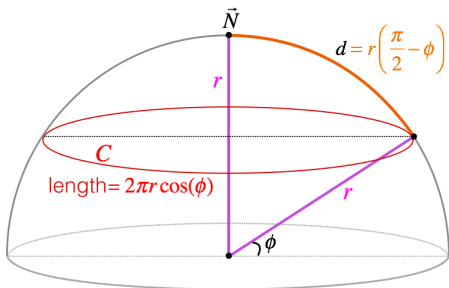
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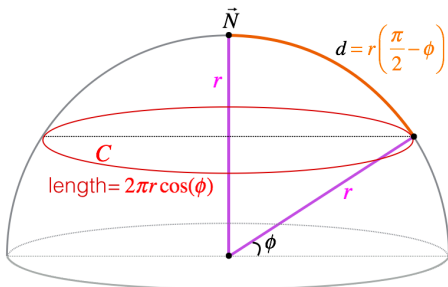
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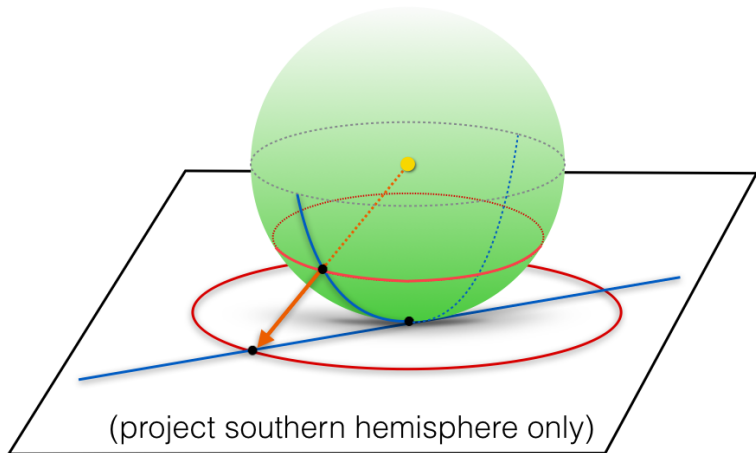
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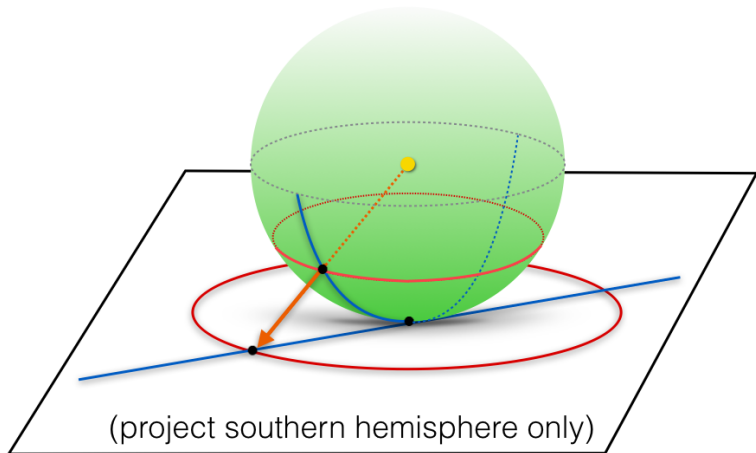
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The gnomonic projection

Gnomonic projection



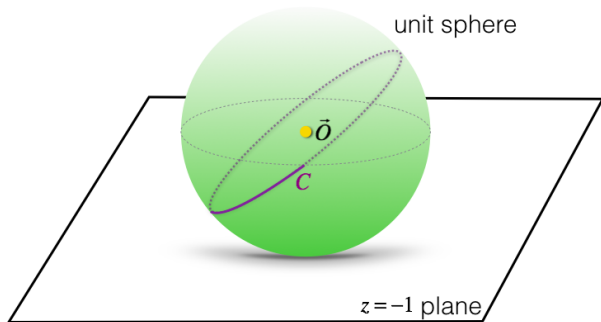
Gnomonic projection



Theorem

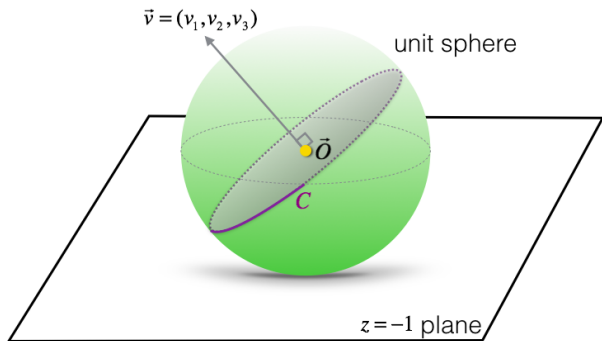
This sends (arcs of) great circles to (segments of) lines.

Great circle arcs map to line segments



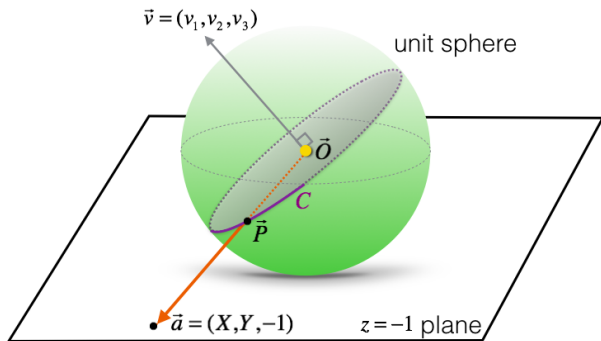
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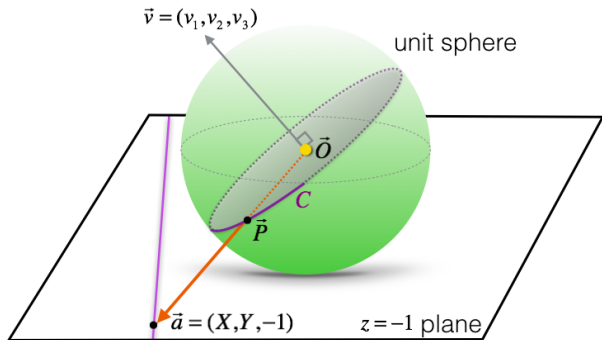
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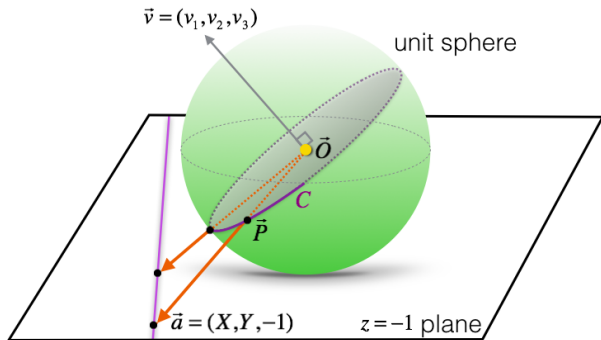
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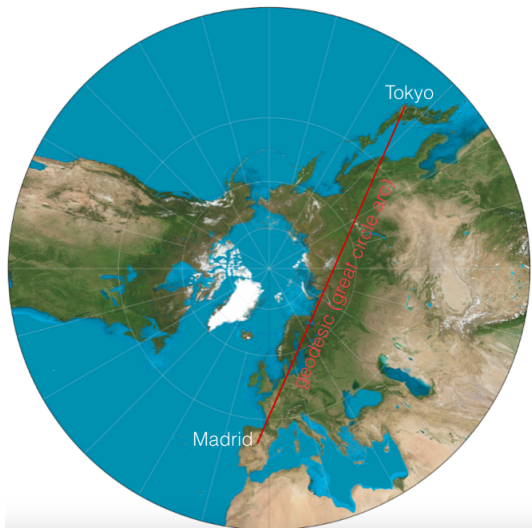
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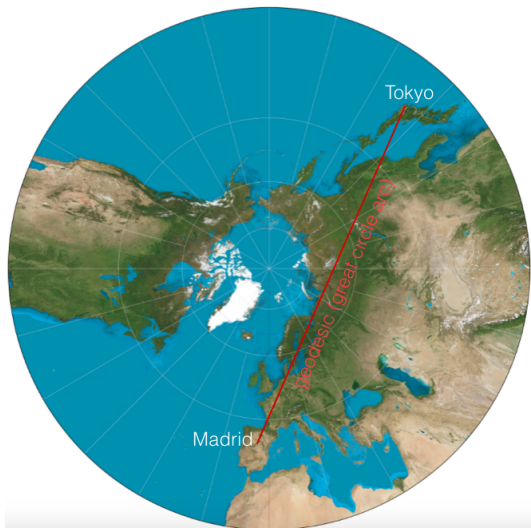
Can check that $f(\phi) = -\cot(\phi)$ does **NOT** satisfy the:

- conformal condition: $f'(\phi) = f(\phi) \sec(\phi)$;
- equi-areal condition: $f'(\phi)f(\phi) = \cos(\phi)$.

Geodesics in the gnomonic map



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Angles aren't preserved, so this is not a good map for navigation!
(Compass bearing changes along a geodesic.)

Measuring distances on the sphere

Length of curves on the sphere

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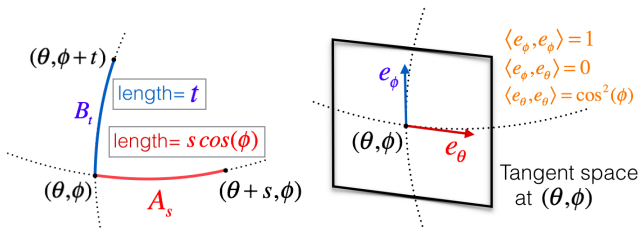
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Remarkably, the circumference of the Earth (hence R) was accurately estimated over 2000 years ago!

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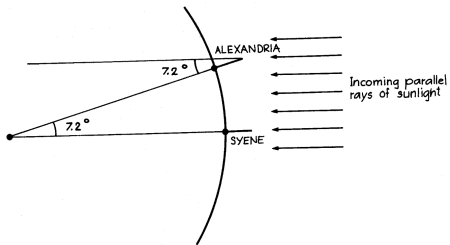
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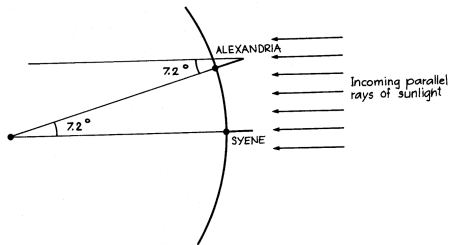
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- He lived in Alexandria, Egypt. There, on June 21 at noon, he looked down a well and could not see the sun. Using a gnomon, he measured the shadow and found the angle of deviation from the vertical = $7.2^\circ = \frac{1}{50}(360^\circ)$.

Circumference of the Earth

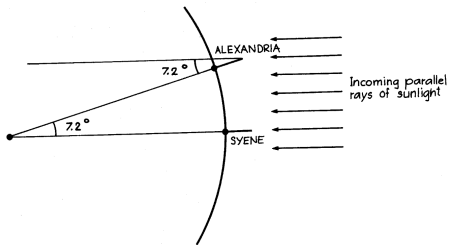


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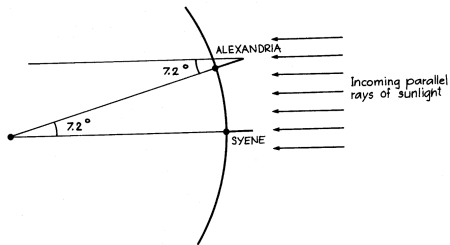
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- From camel caravans travelling up and down the Nile, it was known that $d = 5000$ stadia. Thus, $C = 250,000$ stadia.
- 1 stadion = between 157 m & 185 m. Thus, C is between 39,250 km & 46,250 km. (Pretty good! Actual = 40,075 km.)

MoMS (Mathematics of Maps Seminar) next week:

- **Thursday**, 18 May, 11-12, REALF B302
- Equi-areal maps