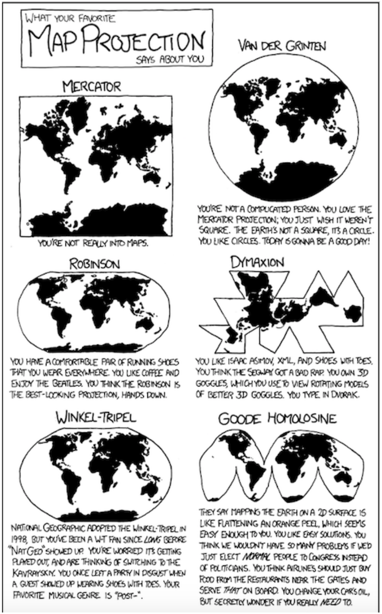
The Mathematics of Maps – Lecture 2

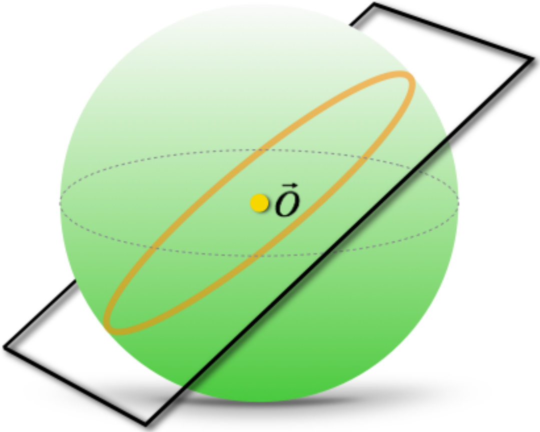


(Source: ▸ xkcd comics)

All maps must lie

Great circles

A great circle is the intersection of the sphere with a plane passing through the sphere's center. These are geodesics on the sphere.



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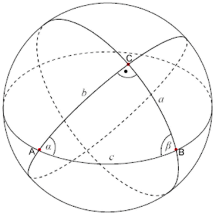

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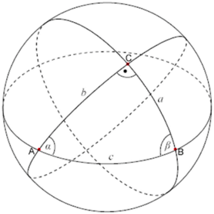

Theorem (Area of a spherical triangle)

Let ΔABC be a spherical triangle (i.e. edges are great circle arcs) on a sphere of radius r. If Σ is its internal angle sum, then

$$0 < Area(\Delta ABC) = (\Sigma - \pi)r^2.$$

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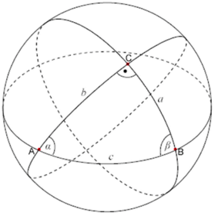

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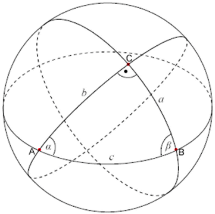

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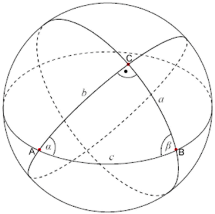

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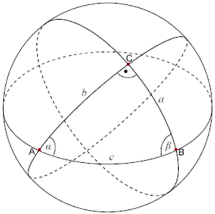

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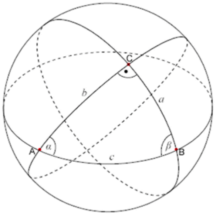

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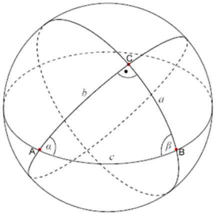

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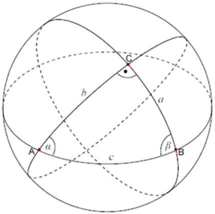

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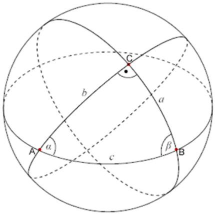

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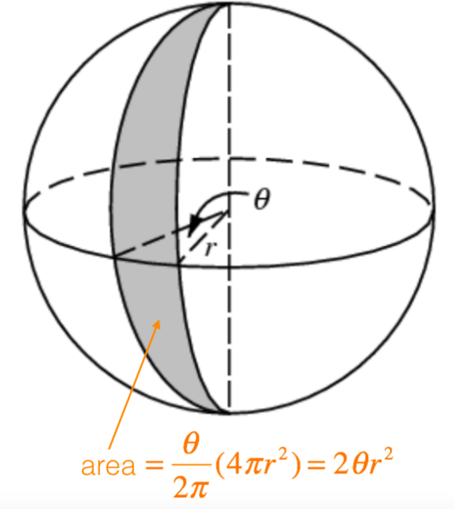
Q: Where does the above AST theorem come from?

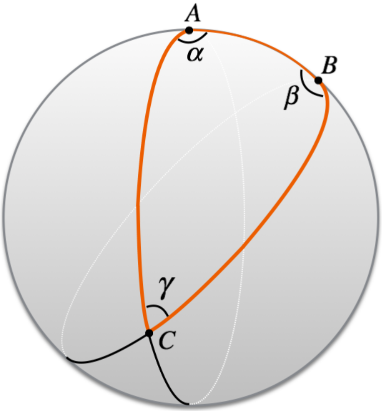
Lunes

Lune = portion between two intersecting great circles and the antipodal points where they cross.



Understanding the AST theorem

Given a spherical ΔABC, rotate so that arc AB looks as below.



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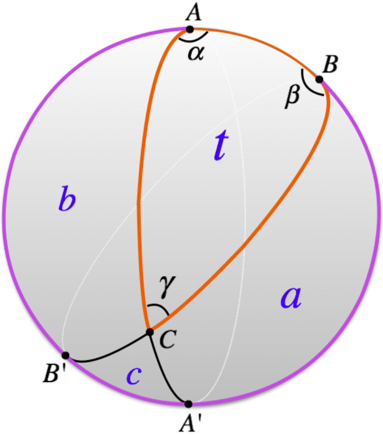

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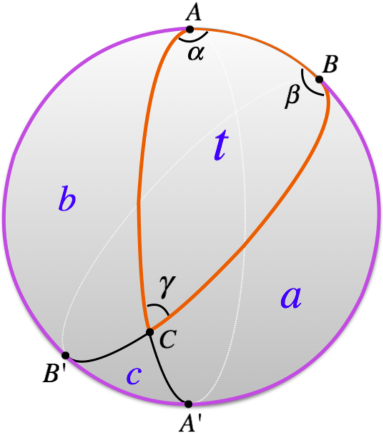

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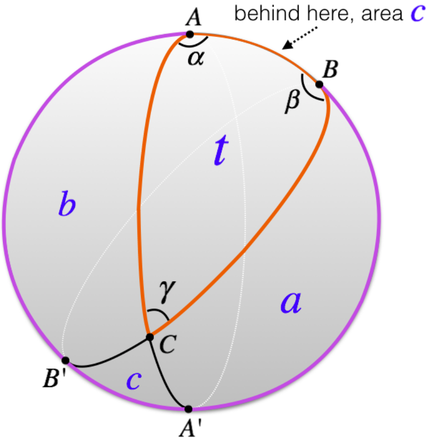

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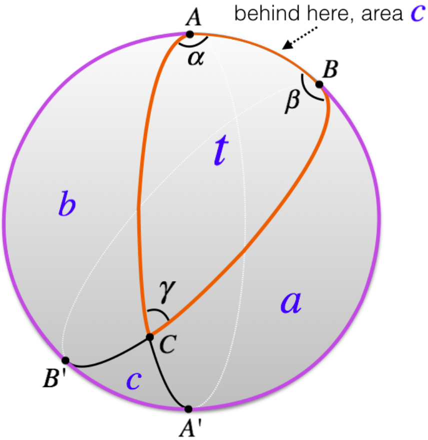

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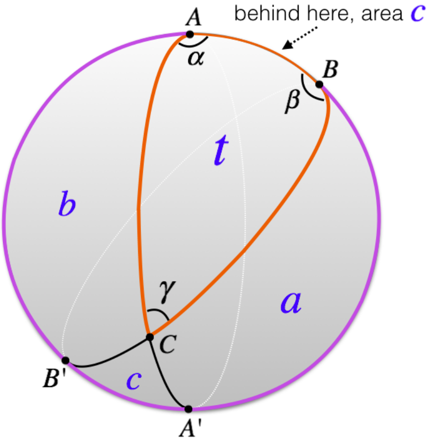

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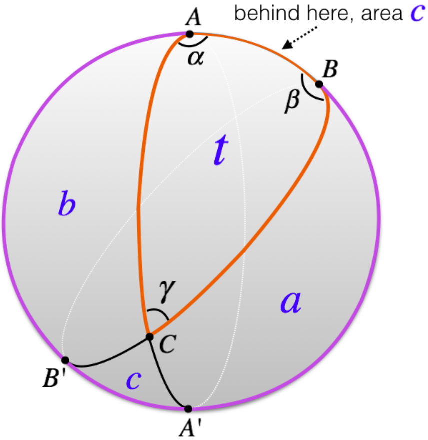

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Subtract & ÷2: $\boxed{t = (\Sigma - \pi) r^2}$. ∴ AST✓

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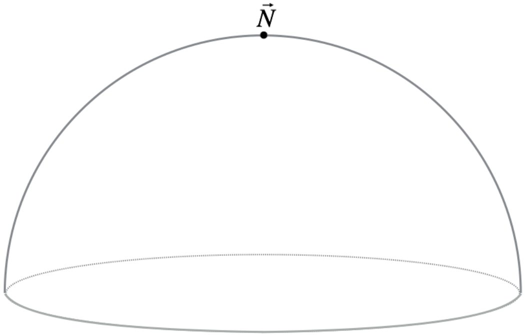

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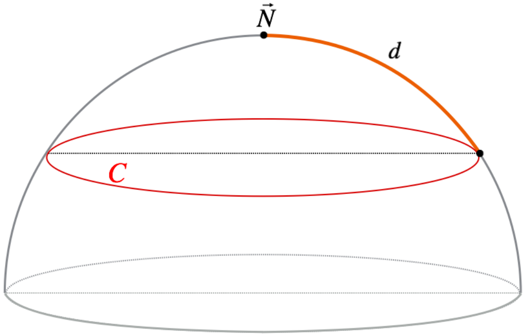

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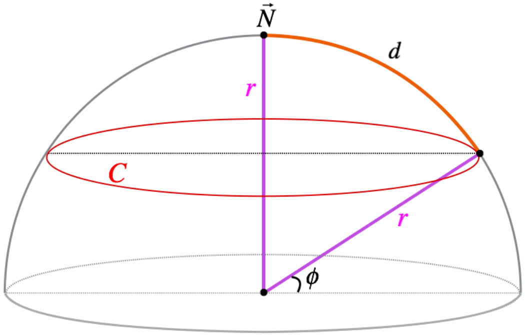

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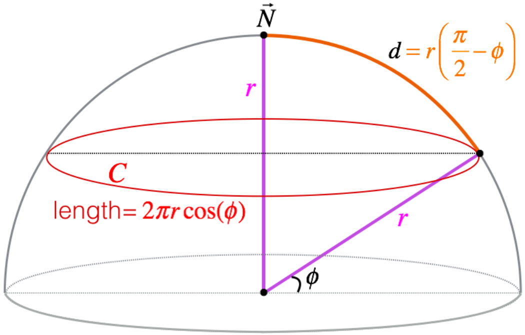

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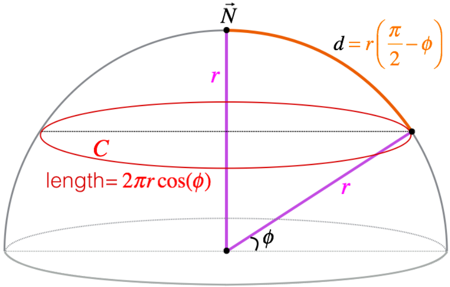

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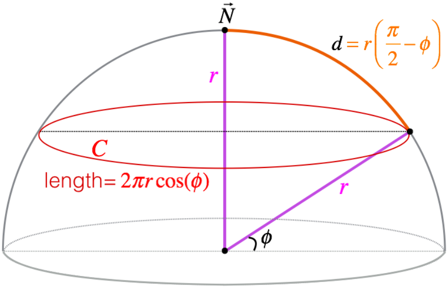

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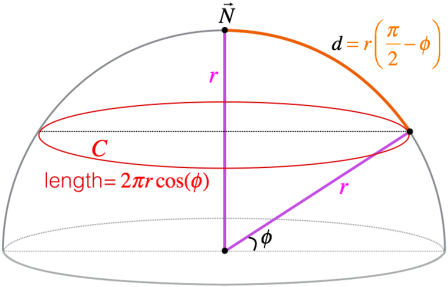

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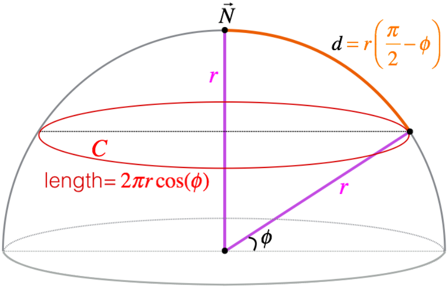

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using the fact that $\sin(x) < x$ for any $x > 0$. (Exercise.)

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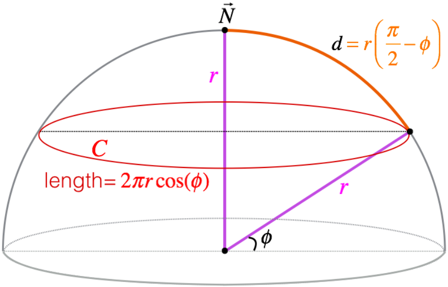

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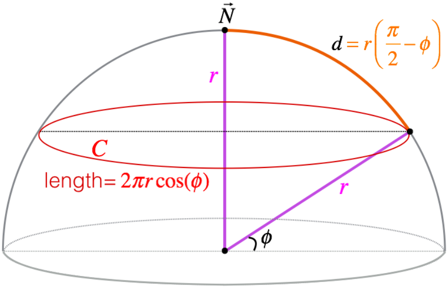

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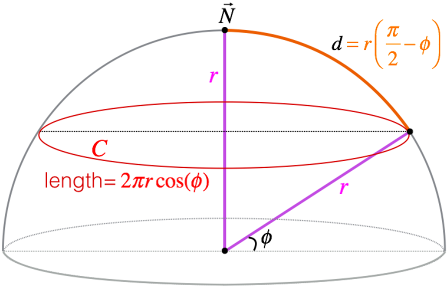

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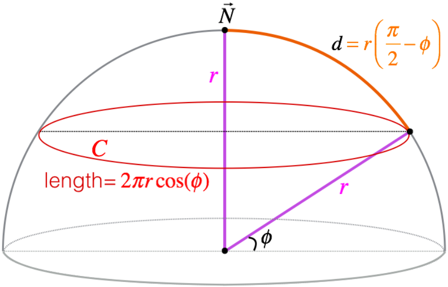

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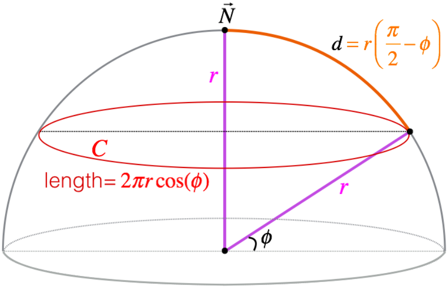

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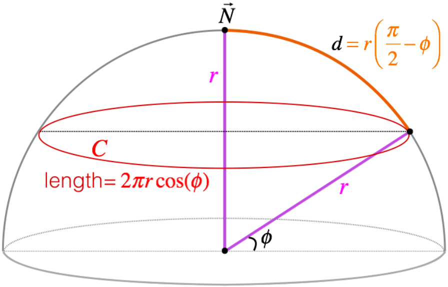

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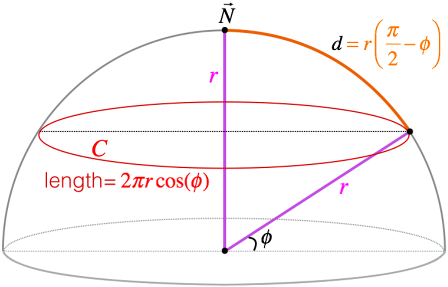

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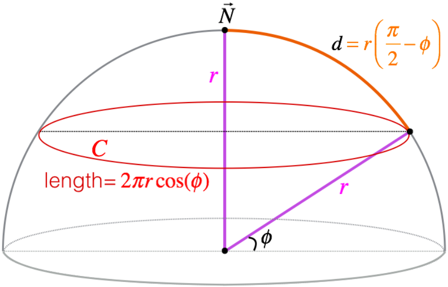

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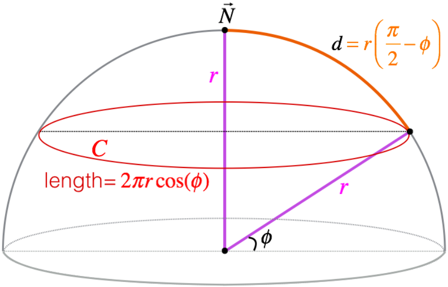
Euler's proof - 2



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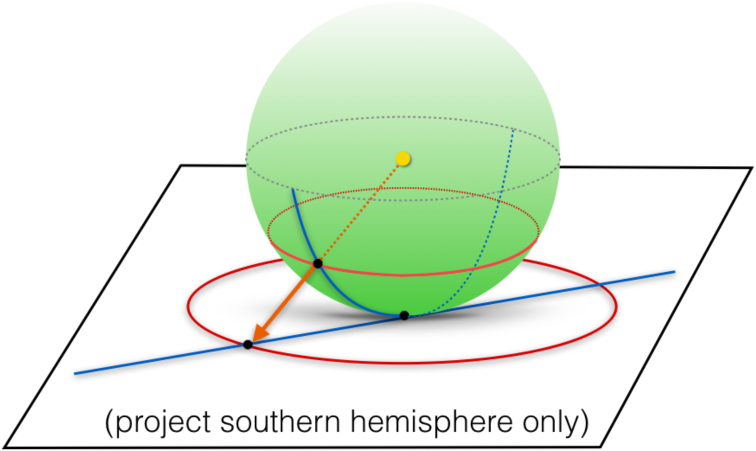
The gnomonic projection

Gnomonic projection



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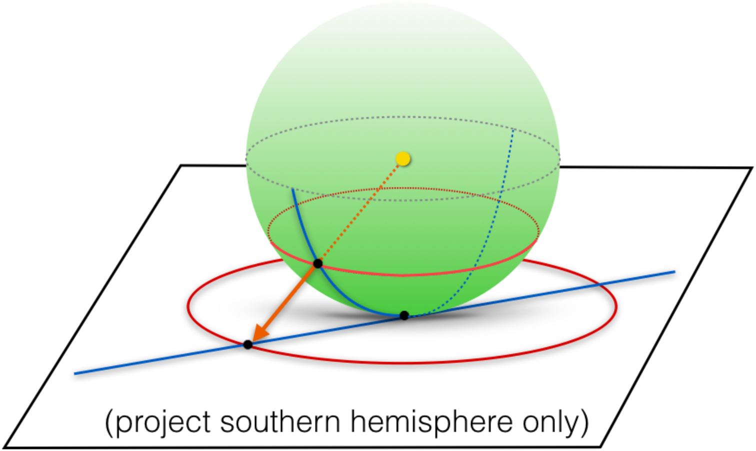
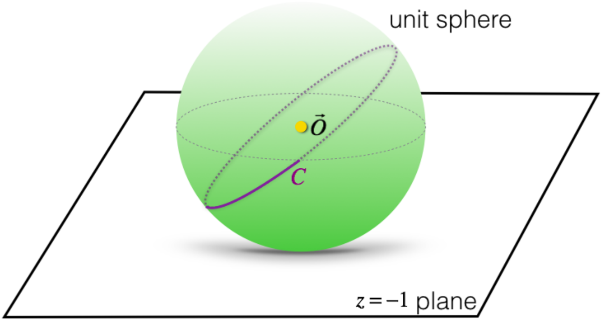


Theorem

This sends (arcs of) great circles to (segments of) lines.

Great circle arcs map to line segments



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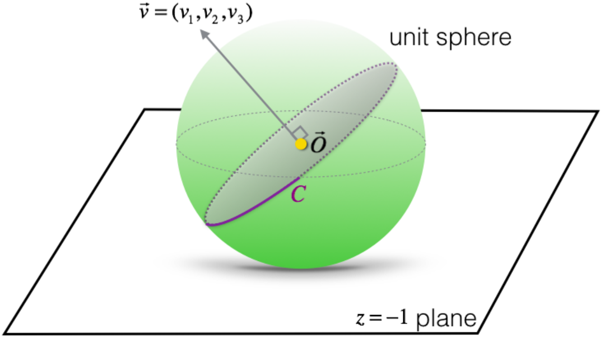

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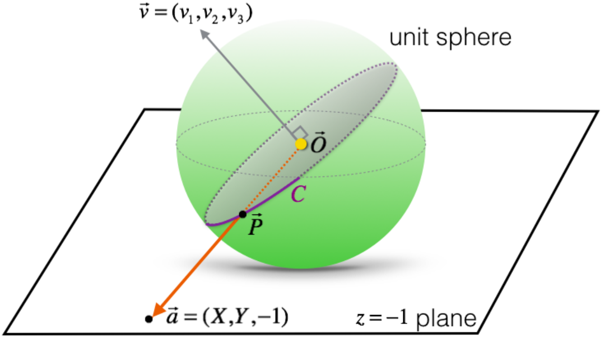

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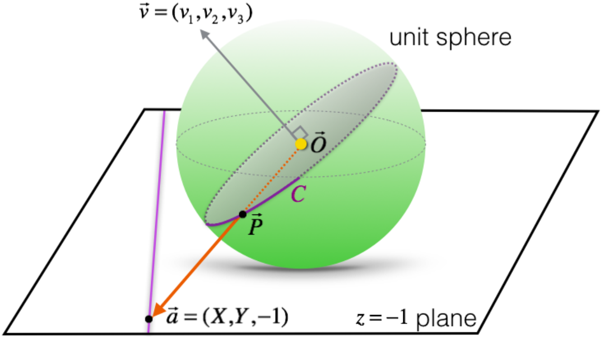

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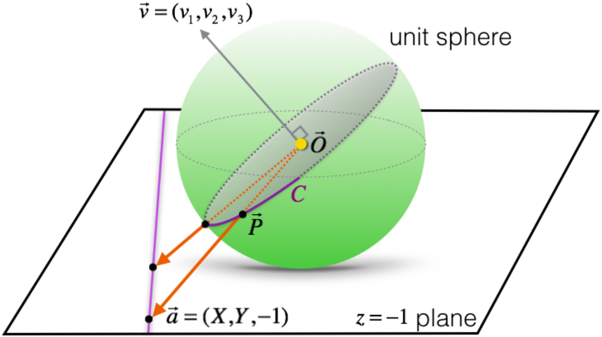

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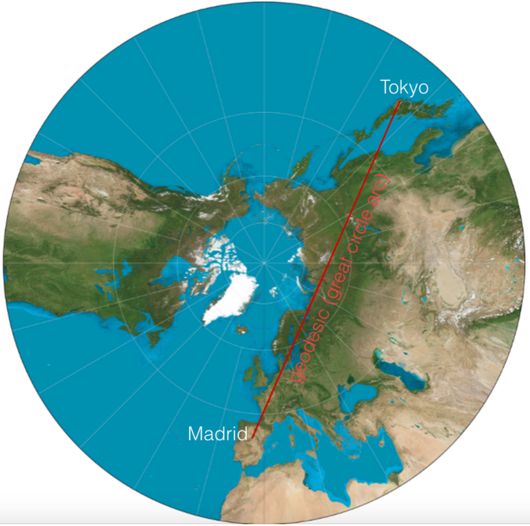
Can check that $f(\phi) = -\cot(\phi)$ does NOT satisfy the:

- conformal condition: $f'(\phi) = f(\phi)\sec(\phi)$;
- equi-areal condition: $f'(\phi)f(\phi) = \cos(\phi)$.

Geodesics in the gnomonic map



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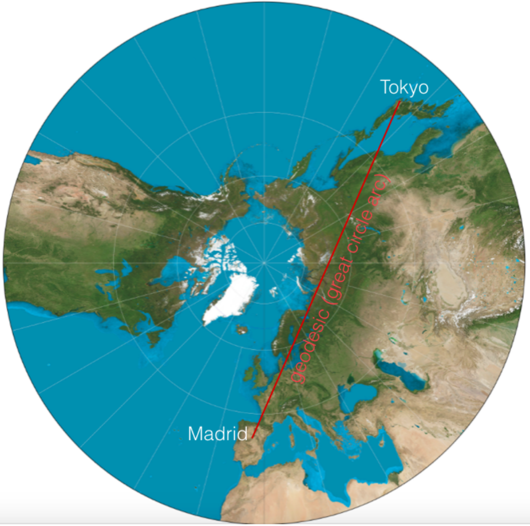

Angles aren't preserved, so this is not a good map for navigation!
(Compass bearing changes along a geodesic.)

Measuring distances on the sphere

Length of curves on the sphere

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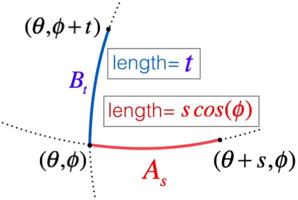

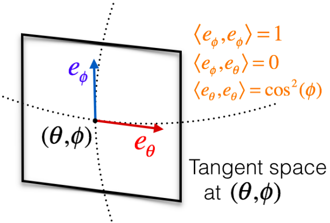

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Remarkably, the circumference of the Earth (hence R) was accurately estimated over 2000 years ago!

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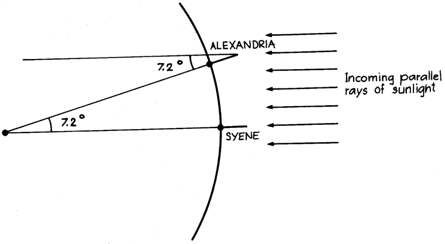
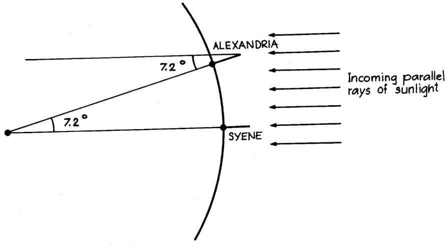
- He lived in Alexandria, Egypt. There, on June 21 at noon, he looked down a well and could not see the sun. Using a gnomon, he measured the shadow and found the angle of deviation from the vertical $= 7.2^\circ = \frac{1}{50}(360^\circ)$.

Circumference of the Earth



Circumference of the Earth



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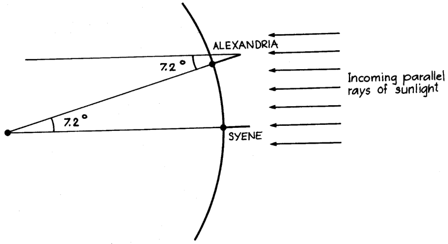

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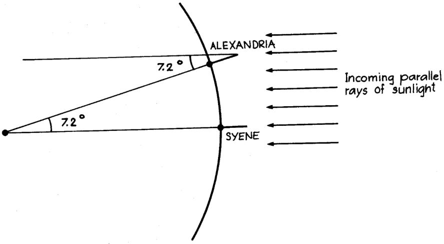

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- 1 stadion = between 157 m & 185 m. Thus, C is between 39,250 km & 46,250 km. (Pretty good! Actual = 40,075 km.)

MoMS will continue...

MoMS (Mathematics of Maps Seminar) next week:

- Thursday, 18 May, 11-12, REALF B302
- Equi-areal maps