The Mathematics of Maps – Lecture 3

(A map hanging in my son's kindergarten class a few years ago.)

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What about Africa? It's enormous... ▸ The True Size of Africa

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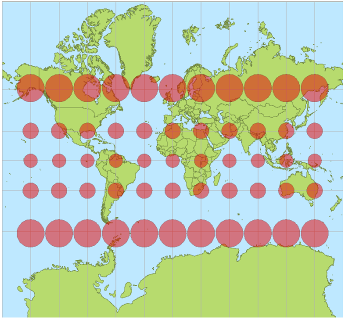

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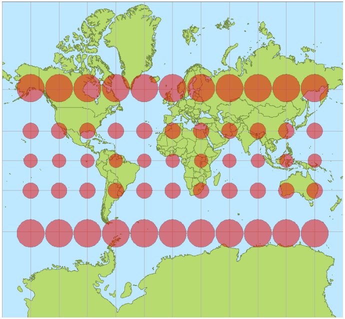

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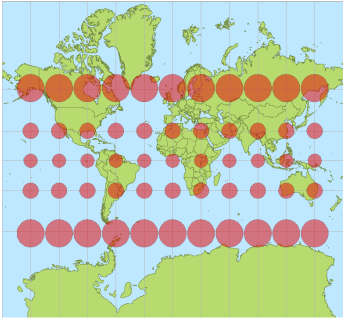

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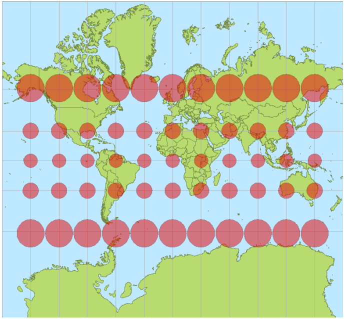

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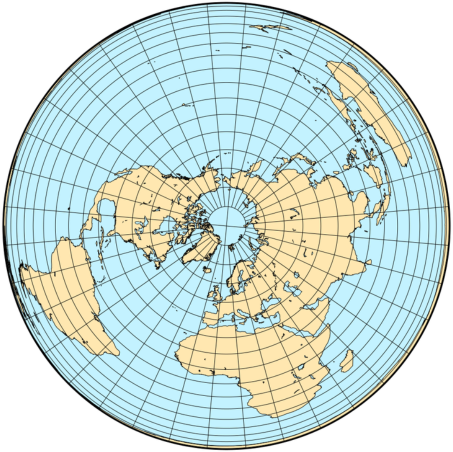
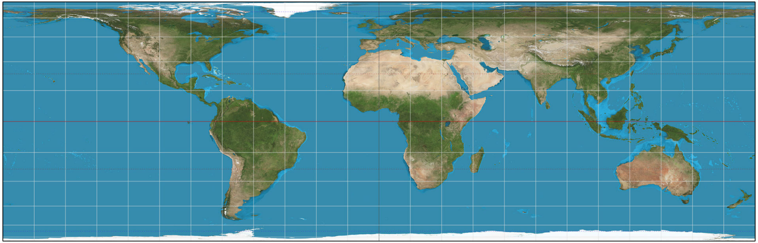
- Equi-areal: $\lambda_m \lambda_p = 1$: indicatrices have same area

Equi-areal maps

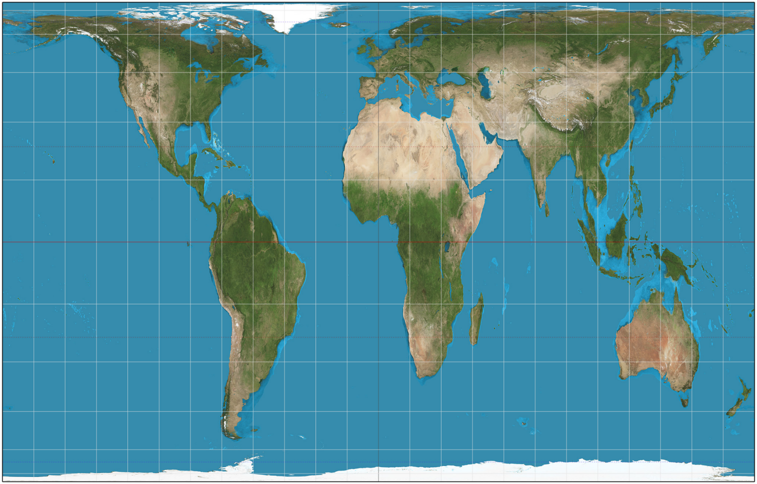
Lambert azimuthal projection



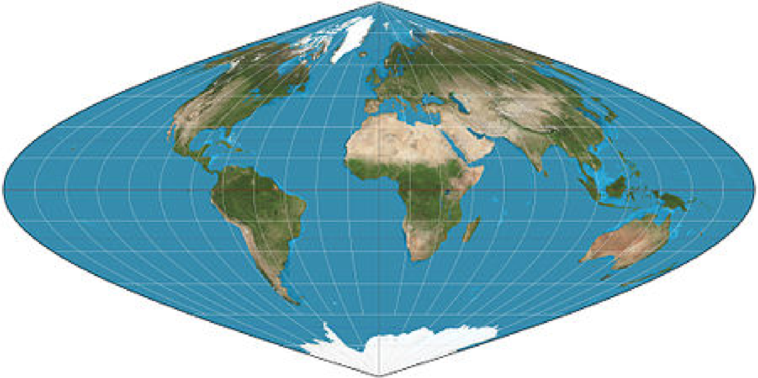
Lambert cylindrical (Archimedes) projection



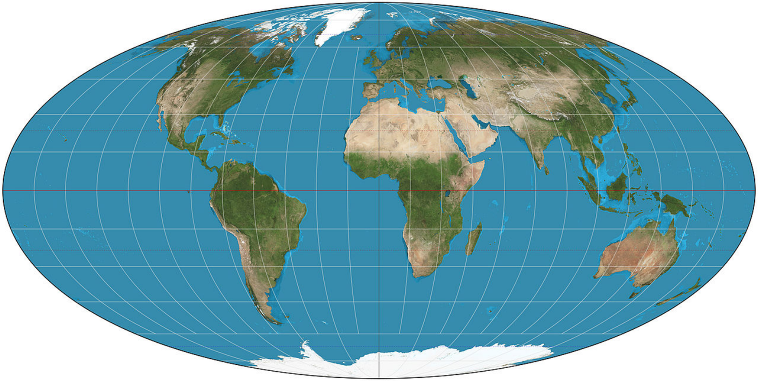
Gall–Peters projection



Sinusoidal projection



Mollweide projection



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On the unit sphere, we have spherical polar coordinates

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$$A(\mathcal{D}) = \int_{\theta_1}^{\theta_2} \int_{\phi_1}^{\phi_2} \cos\phi\, d\phi\, d\theta = (\theta_2 - \theta_1)(\sin\phi_2 - \sin\phi_1) \quad (0.1)$$

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Full sphere: take $\theta_1 = -\pi$, $\theta_2 = \pi$, $\phi_1 = -\frac{\pi}{2}$, $\phi_2 = \frac{\pi}{2}$ $\rightsquigarrow$ $\boxed{A = 4\pi}$.

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Lambert's equi-areal azimuthal projection

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Lambert equi-areal azimuthal - 2

$$f(\phi) = \sqrt{2(1-\sin(\phi))} = \sqrt{2\left(1-\cos\left(\frac{\pi}{2}-\phi\right)\right)}$$
$$= \sqrt{2\left(2\sin^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right)} = 2\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

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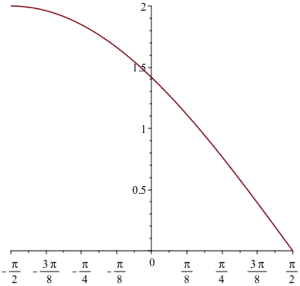
using the double angle formula $\cos(2\alpha) = 1 - 2\sin^2(\alpha)$. Note $f(\phi)$ is positive and strictly decreasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$.



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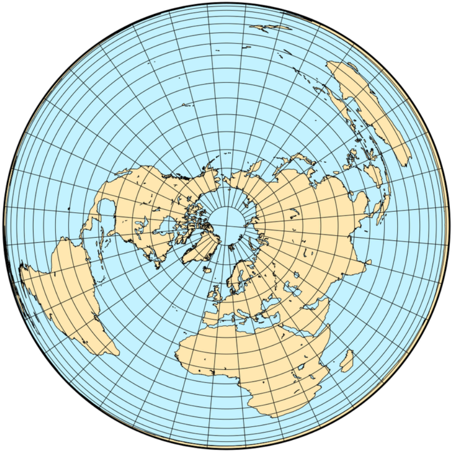
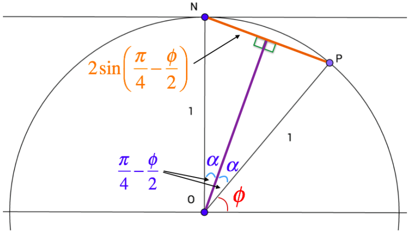
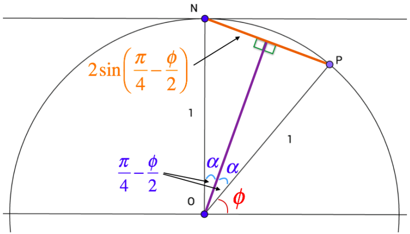


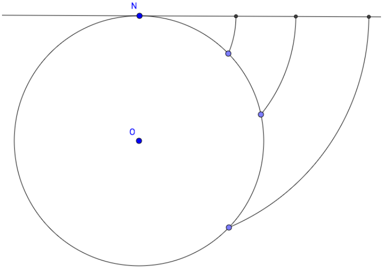
$$\boxed{R = 2\sin\left(\frac{\pi}{4} - \frac{\phi}{2}\right), \quad \Theta = \theta}$$

Lambert equi-areal azimuthal - 4





Projection recipe: Follow circular arcs with centre N through P to the tangent plane at N.



Lambert / Archimedes projection

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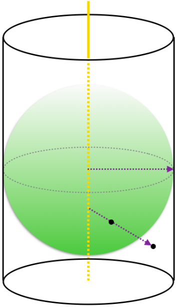

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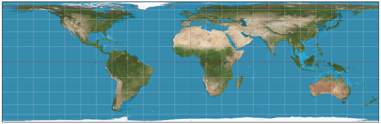

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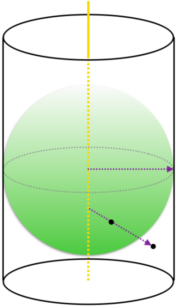

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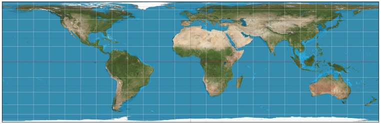

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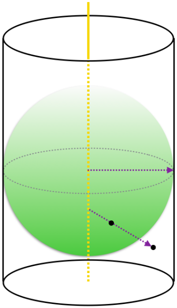

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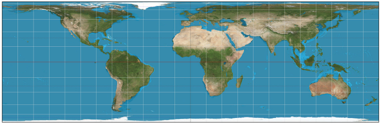

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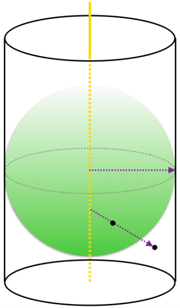

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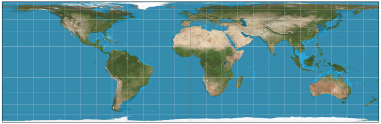

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Solve it to get: $f(\phi) = \sin(\phi)\sec(\phi_0)$.

Gall–Peters - 2

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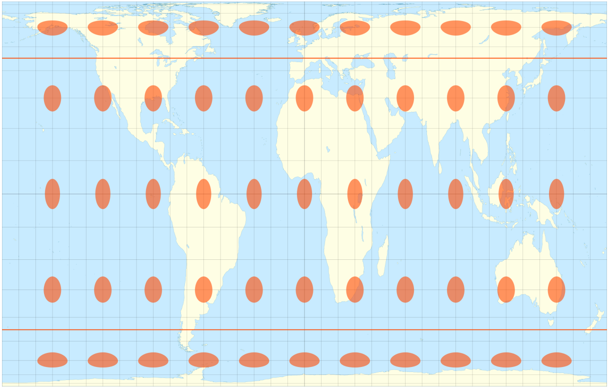
$\phi_0 = \frac{\pi}{4}$: Gall–Peters projection



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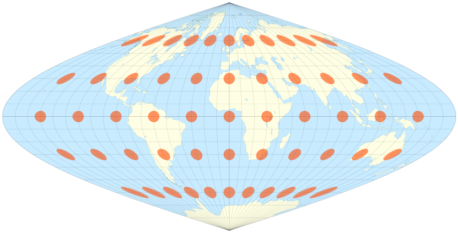
Sinusoidal projection (1570): Discard rectangular map shape.



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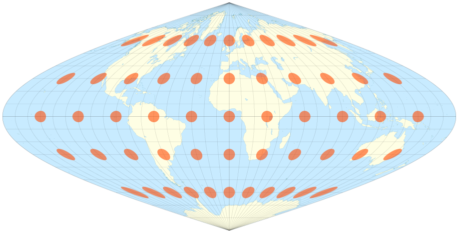

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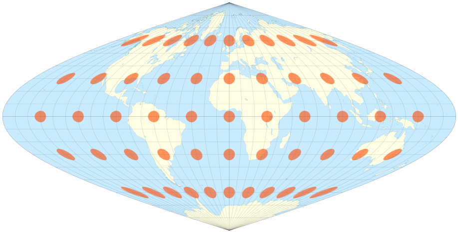

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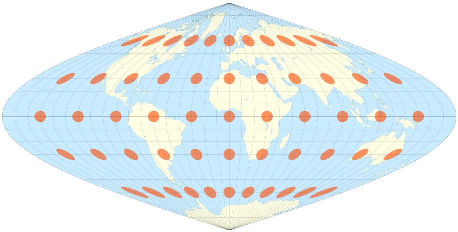

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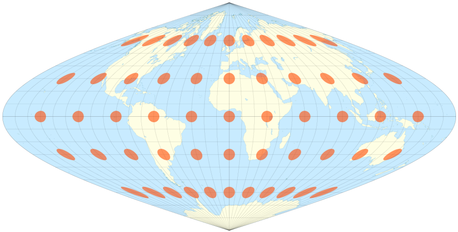

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- Images of parallels are evenly spaced.

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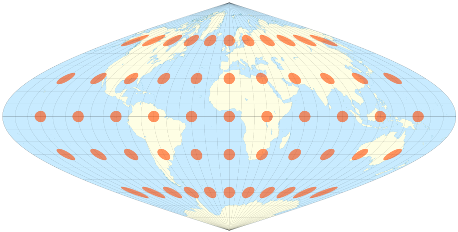

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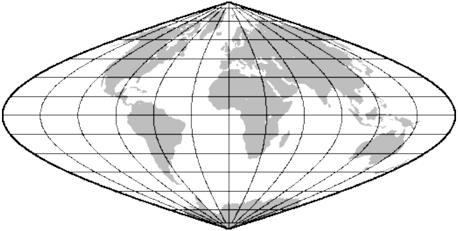
- Parallels $\mapsto$ horizontal line segments in *correct proportion*.
- Images of parallels are evenly spaced.
- Images of meridians are evenly spaced.

Sinusoidal projection - 2



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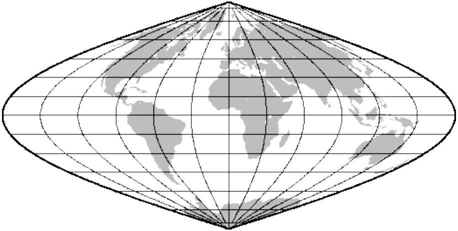

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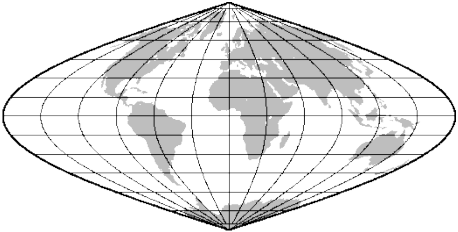

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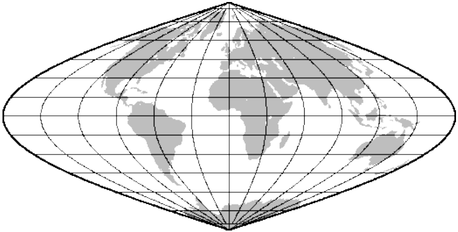

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Sinusoidal projection:

$$\Psi : \quad \boxed{\begin{cases} X = \theta\cos(\phi), & -\pi < \theta < \pi; \\ Y = \phi, & -\frac{\pi}{2} < \phi < \frac{\pi}{2} \end{cases}}$$

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Let $\mathcal{D}$ = region btw latitudes $\phi_1 \leq \phi_2$ & longitudes $\theta_1 \leq \theta_2$.

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Thus, the map is equi-areal.

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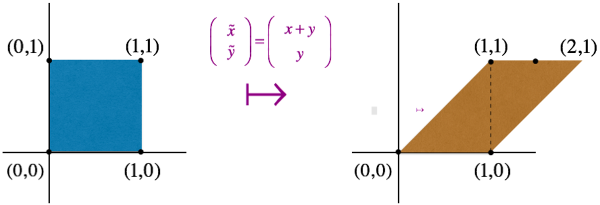
Instead of $dXdY$, think of area element instead as $dX \wedge dY$.

$$dX \wedge dY = (-\theta \sin\phi d\phi + \cos(\phi) d\theta) \wedge d\phi = \cos(\phi) d\theta \wedge d\phi,$$

i.e. area element for the plane becomes area element for S^2.

The following linear map is area-preserving:



Area-preserving linear transformations

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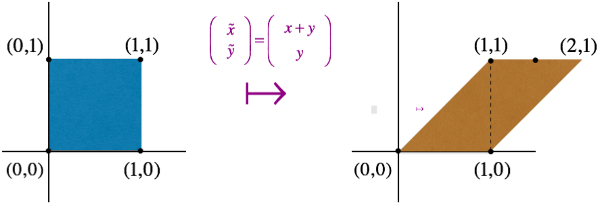

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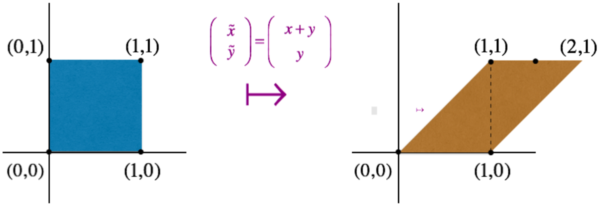

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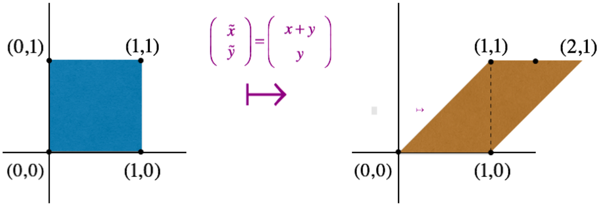

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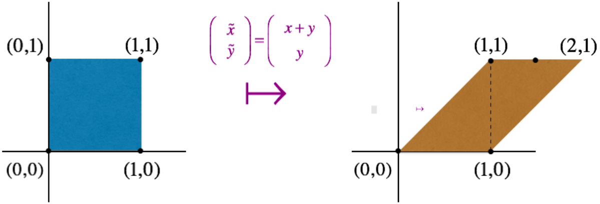

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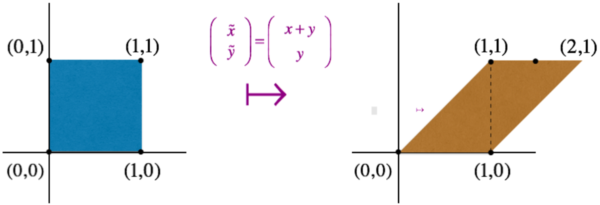

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A: $\boxed{det = 1}$ matrices!

Mollweide projection

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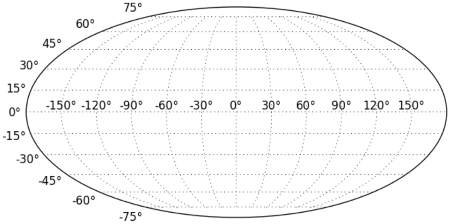

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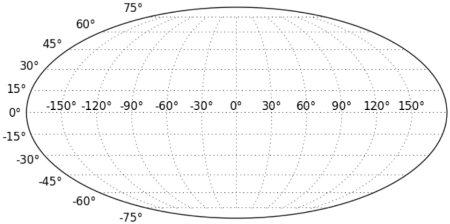
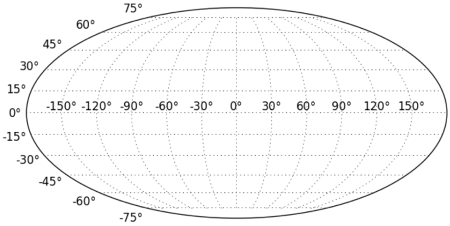


- Equi-areal

Mollweide projection



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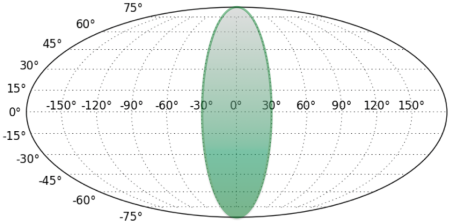

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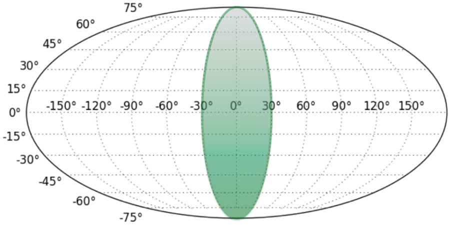

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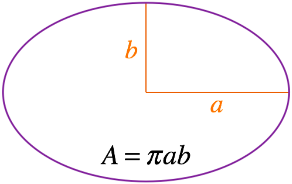
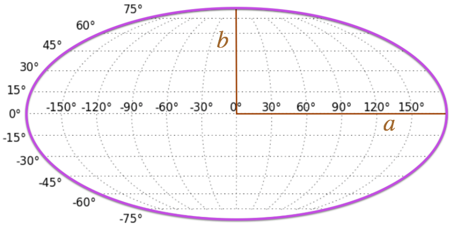
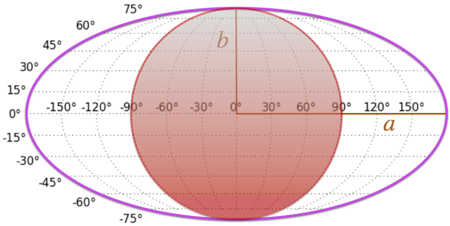


Mollweide - 2



- $E = E_\pi$: Equi-areal $\Rightarrow$ $Area(S^2) = 4\pi = Area(E)$.



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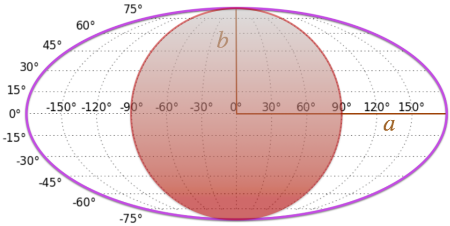

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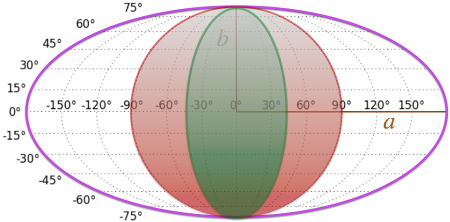

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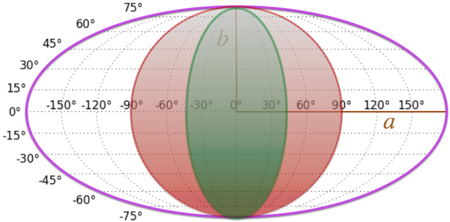

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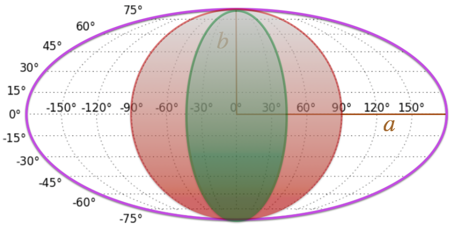

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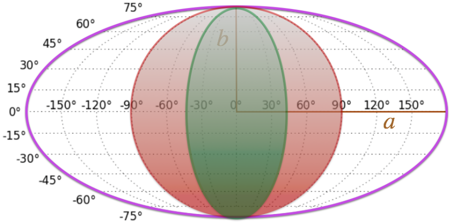

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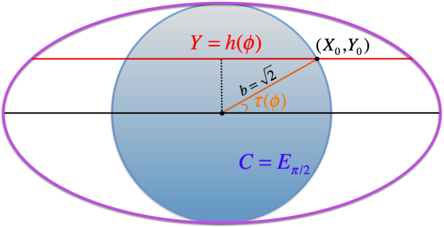
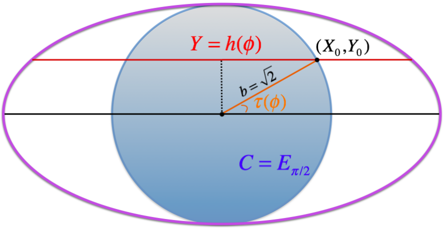
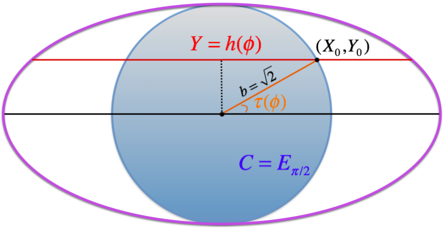
But what does the map function look like?


$Y = h(\phi)$
(X_0, Y_0)
$b = \sqrt{2}$
$\tau(\phi)$
$C = E_{\pi/2}$



On C, we have $X_0 = \sqrt{2}\cos(\tau(\phi))$ and $Y_0 = h(\phi) = \sqrt{2}\sin(\tau(\phi))$.

Mollweide - 3



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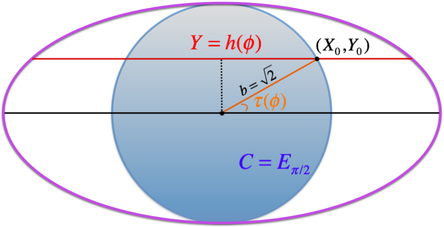

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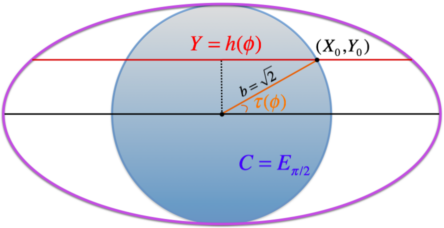

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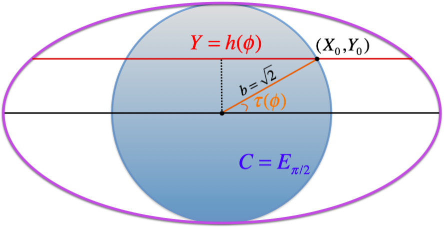

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$\therefore$ Strip btw $Y = 0$ and $Y = h(\phi)$ has area $2\pi\sin\phi$. $\therefore$ Strip inside C has area $\pi\sin\phi = \sqrt{2}\cos(\tau)h + 2\frac{\tau}{2\pi}\pi(\sqrt{2})^2 = \sin(2\tau) + 2\tau$.
This defines $\tau(\phi)$ implicitly.

Mollweide - 4

Check: $dX \wedge dY = \cos\phi \, d\theta \wedge d\phi$. (Identity $\pi \cos\phi = 4\cos^2(\tau)\tau'$ follows from differentiating $\boxed{\pi \sin\phi = \sin(2\tau) + 2\tau}$ $(*)$)

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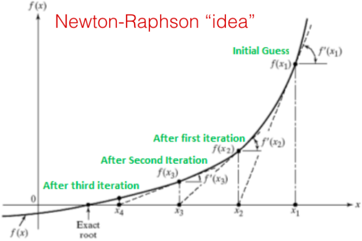

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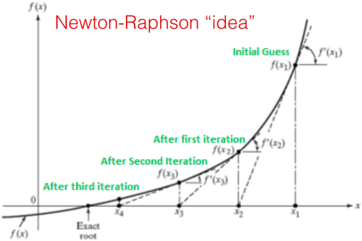

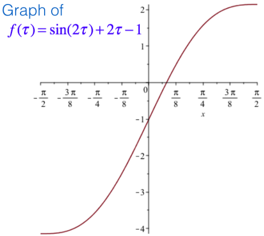

Fix ϕ_0. Define $f(\tau) = \sin(2\tau) + 2\tau - \pi \sin\phi_0$, $-\frac{\pi}{2} \le \tau \le \frac{\pi}{2}$, and

$$\tau_0 = 0, \quad \tau_{n+1} = \tau_n - \frac{f(\tau_n)}{f'(\tau_n)}.$$

Iterates will converge to a root of $f(\tau) = 0$.

Math Of Maps will conclude...

- Conformal mappings
- Stereographic projection
- Mercator projection
- What does $\mathbb{C}$ have to do with it?