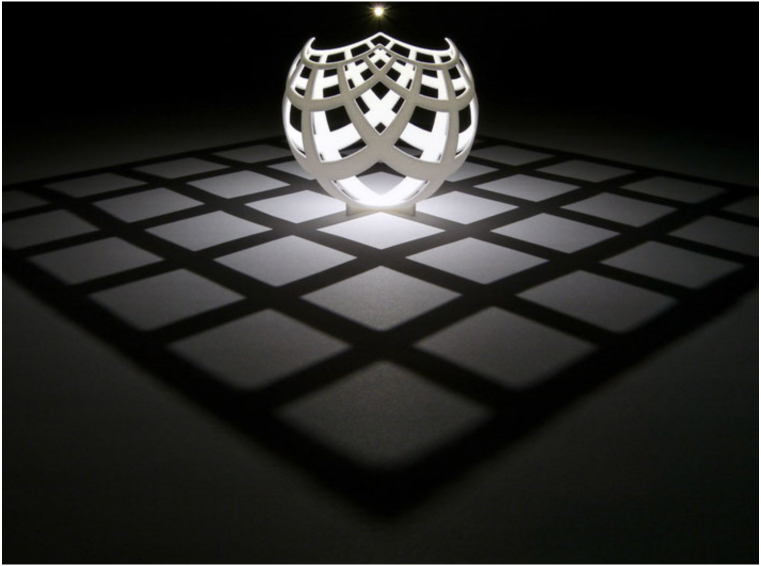
The Mathematics of Maps – Lecture 4



Mercator projection

The Mercator projection (1569)

- Importance: Navigation.

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- Here's a fun little puzzle: ▸ Mercator Puzzle

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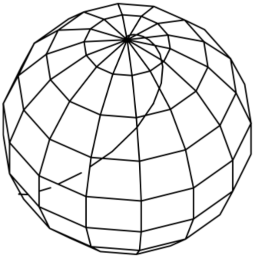
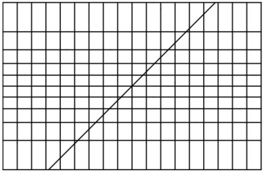


$\mapsto$



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meridians of longitude	$\mapsto$	vertical lines
rhumb lines	$\mapsto$	straight lines

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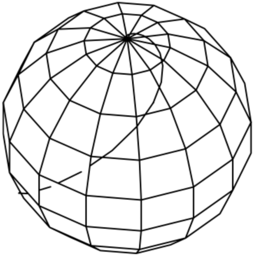

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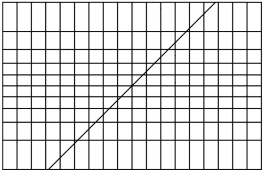

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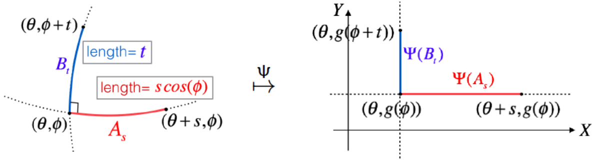
The last one makes the Mercator map useful for navigation.

Mercator map formula

For a map $\boxed{\Psi : \; X = \theta, \; Y = g(\phi)}$, we calculated scale factors:



$$\lambda_p = \lim_{s \to 0} \frac{L(\Psi(A_s))}{L(A_s)} = \sec(\phi), \quad \lambda_m = \lim_{t \to 0} \frac{L(\Psi(B_t))}{L(B_t)} = |g'(\phi)|.$$

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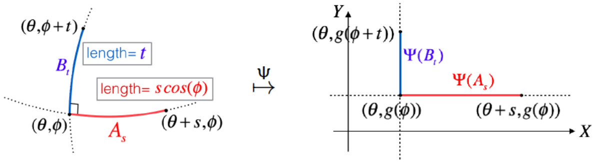

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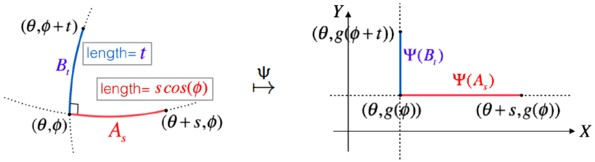

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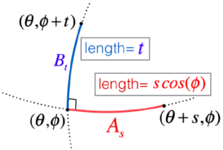

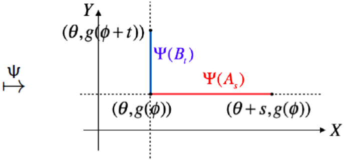

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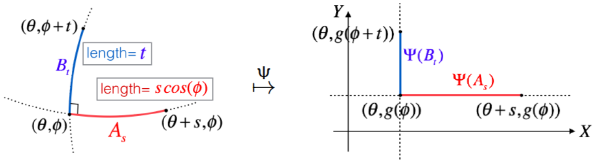

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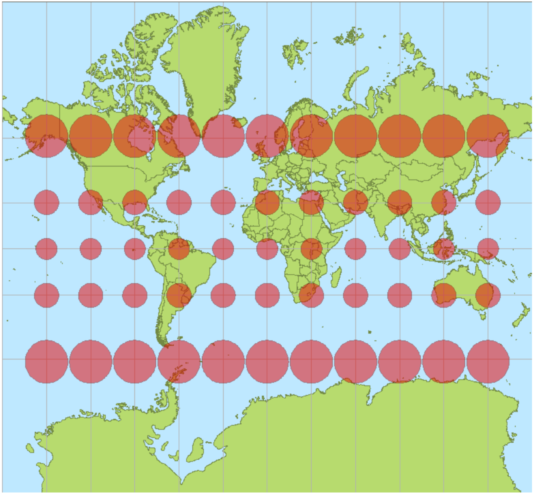
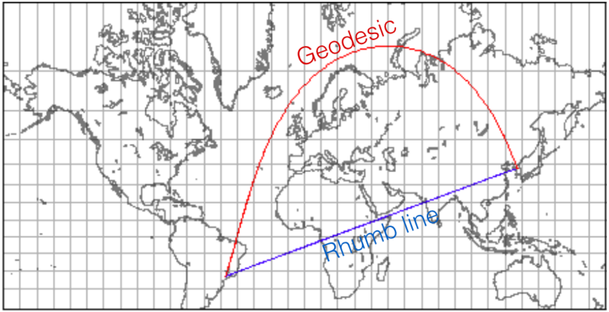
$$\boxed{\Psi_M : \quad \begin{cases} X = \theta, & -\pi \leq \theta \leq \pi \\ Y = \ln(\sec\phi + \tan\phi), & -\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2} \end{cases}.}$$

Distortion in the Mercator projection



Distance along a rhumb line

Q: How can we find the distance along a rhumb line?



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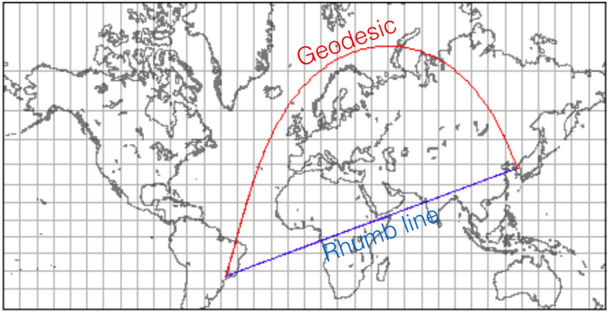

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For a path $(\phi(t), \theta(t))$, $0 \leq t \leq 1$ on S^2, we saw in Lecture 2 a formula for its length $\boxed{L = \int_0^1 \sqrt{(\phi')^2 + \cos^2(\phi)(\theta')^2} dt}$.

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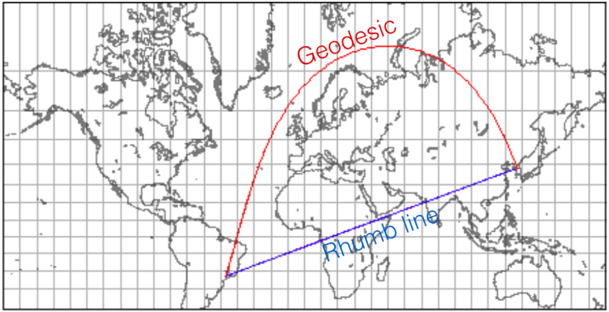

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Need ϕ, θ as functions of X, Y, i.e. inverse Mercator map Ψ_M^{-1}.

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Using some trig. trickery... $e^Y = \sec(\phi) + \tan(\phi) = \frac{1+\sin(\phi)}{\cos(\phi)} = \frac{1-\cos(\phi+\frac{\pi}{2})}{\sin(\phi+\frac{\pi}{2})} = \frac{2\sin^2(\frac{\phi}{2}+\frac{\pi}{4})}{2\sin(\frac{\phi}{2}+\frac{\pi}{4})\cos(\frac{\phi}{2}+\frac{\pi}{4})} = \tan(\frac{\phi}{2} + \frac{\pi}{4})$.

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$$\Psi_M^{-1} : \quad \begin{cases} \theta = X \\ \phi = 2\arctan(e^Y) - \frac{\pi}{2} \end{cases}$$

Distance along a rhumb line - 2

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(When $c \to 0$, we recover $L = \frac{2e^d}{1+e^{2d}} = \cos(\phi)$.) This formula is valid on the unit sphere S^2. On a sphere of radius R, this length would be rescaled by R.

Stereographic projection

Stereographic projection (from north pole)

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parallels of latitude	$\mapsto$	circles centred at 0
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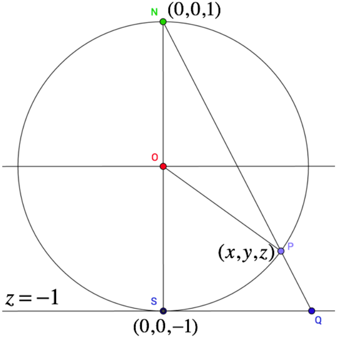
(Note: Viewed from above, East $\rightsquigarrow$ West is a clockwise rotation in the map.)

Geometric construction

On S^2, put a light source at the north pole, and cast the shadow of $\vec{P} \in S^2$ onto the plane tangent at the south pole.



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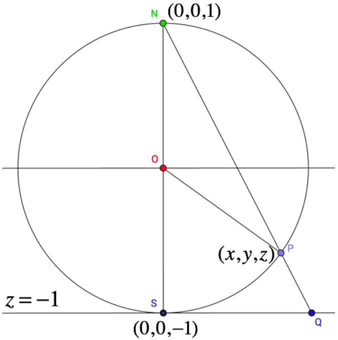

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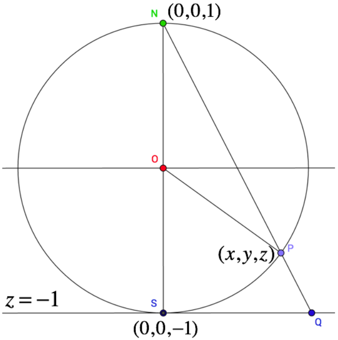

Line $\vec{NP}$ intersects $z = -1$ plane at $\vec{Q}$: $\left(\frac{2x}{1-z}, \frac{2y}{1-z}, -1\right)$. Hence,

$$\boxed{\Psi_N : \quad X = \frac{2x}{1-z}, \quad Y = \frac{2y}{1-z}}.$$

Polar coordinate formula

Use spherical polar coords $(x, y, z) = (\cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi)$ to write Ψ_N.

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Towards north pole, $\lim\limits_{\phi \to \frac{\pi}{2}^-} R \to \infty$. (High distortion.)

Towards south pole, $\lim\limits_{\phi \to -\frac{\pi}{2}^+} R \to 0$.

Stereographic projection is conformal

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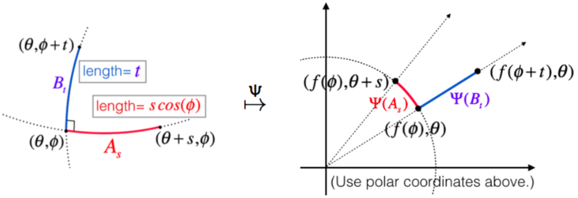

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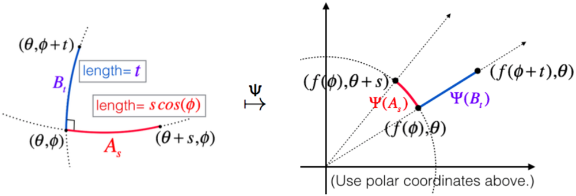

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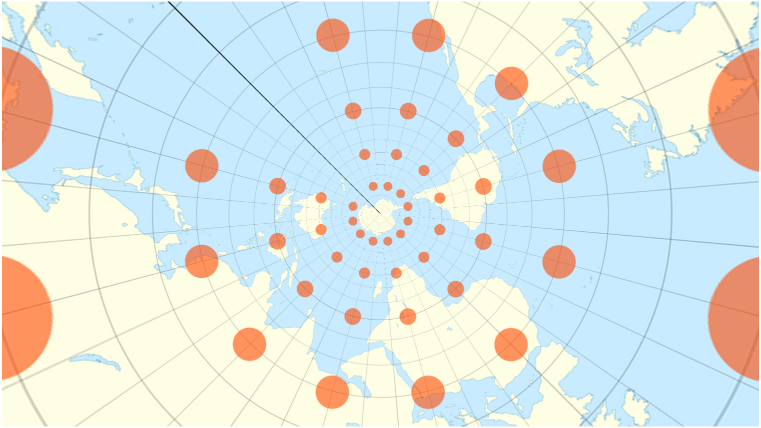
Conformality means $\boxed{\lambda_p = \lambda_m}$. For $f(\phi) = 2(\sec\phi + \tan\phi)$, we can verify this is true. (Note $f'(\phi) > 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$.)

Distortion in the stereographic projection



Circles are mapped to “circles”

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A circle *on* S^2 *is the intersection of a plane* Π *in* $\mathbb{R}^3$ *with* S^2.
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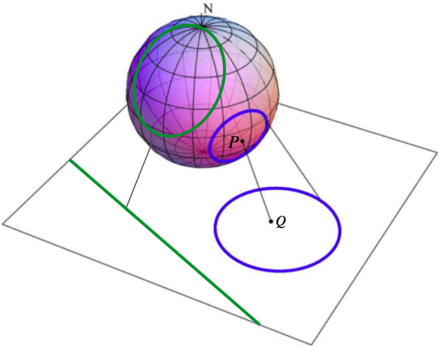

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IDEA: Invert $\Psi_N: \ X = \frac{2x}{1-z}, Y = \frac{2y}{1-z}$ to get

$$P = (x, y, z) = \left(\frac{4X}{X^2 + Y^2 + 4}, \frac{4Y}{X^2 + Y^2 + 4}, \frac{X^2 + Y^2 - 4}{X^2 + Y^2 + 4} \right) \in S^2.$$

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$$4(aX + bY) + c(X^2 + Y^2 - 4) = d(X^2 + Y^2 + 4)$$

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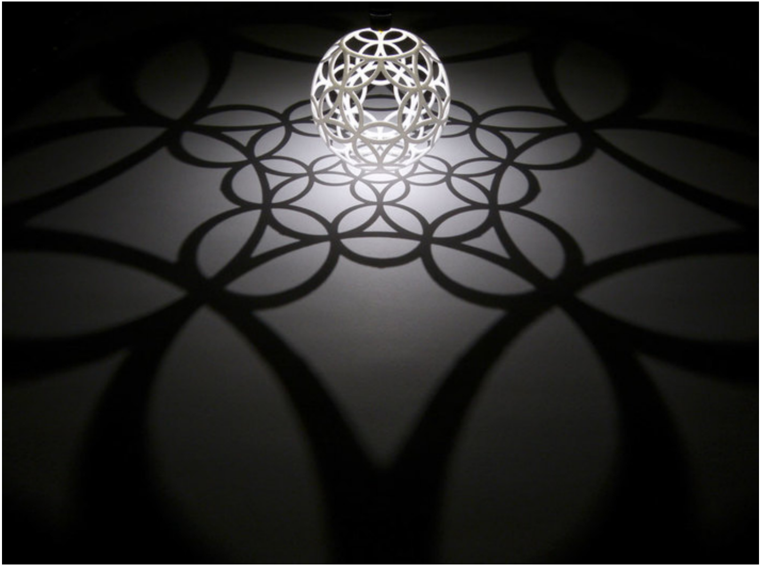
$$4(aX + bY) + c(X^2 + Y^2 - 4) = d(X^2 + Y^2 + 4)$$

This is the eqn of a circle if $c \neq d$ and a line if $c = d$.

Circles images



From Mercator to stereographic projection

A simple complex relationship

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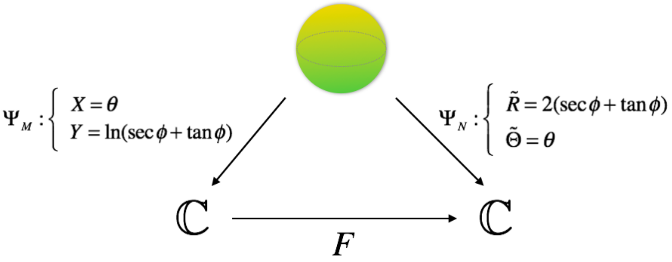

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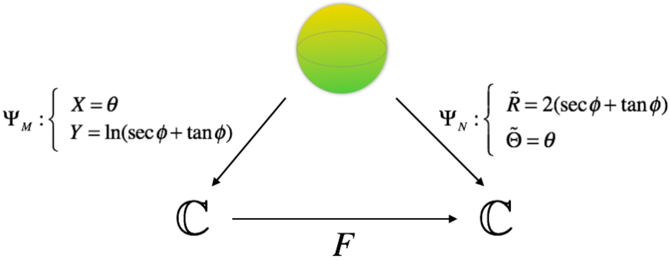

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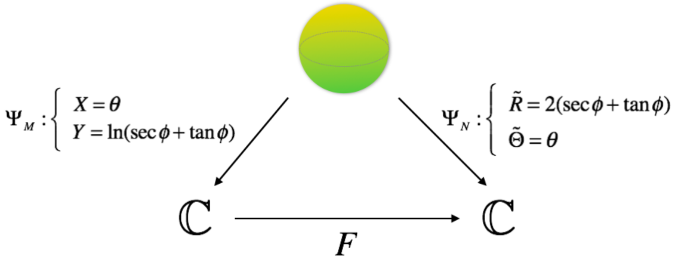

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Using the *complex* exponential, we have

$$\tilde{Z} = \tilde{R}e^{i\tilde{\Theta}} = 2e^{Y}e^{iX} = 2e^{i(X-iY)} = 2e^{i\overline{Z}}.$$

Complex differentiable functions

Given $F : \mathbb{C} \to \mathbb{C}$, define $\mathbb{C}$-differentiability via the usual formula

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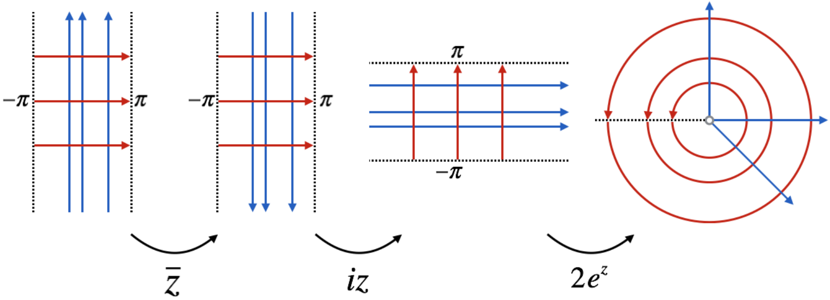
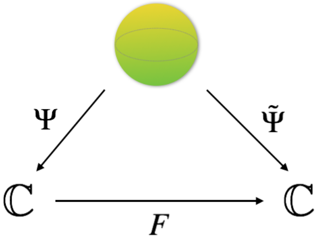
The Mercator-to-stereographic map is conformal and decomposes:



New conformal maps from old

Given a conformal map Ψ from the sphere to the plane, we can use a $\mathbb{C}$-differentiable F (with $F' \neq 0$) to produce a new map $\tilde{\Psi}$:



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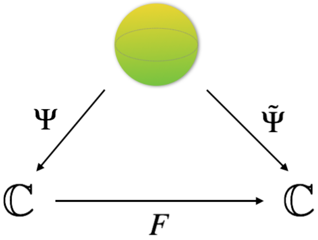

Möbius transformations $F(z) = \frac{az+b}{cz+d}$ form a wonderful class of $\mathbb{C}$-differentiable (hence conformal) maps. ▸ Möbius transformations revealed

A family of conformal maps

Here's another way from Mercator to stereographic projection:

▸ The Mercator and stereographic projections, and many in between .

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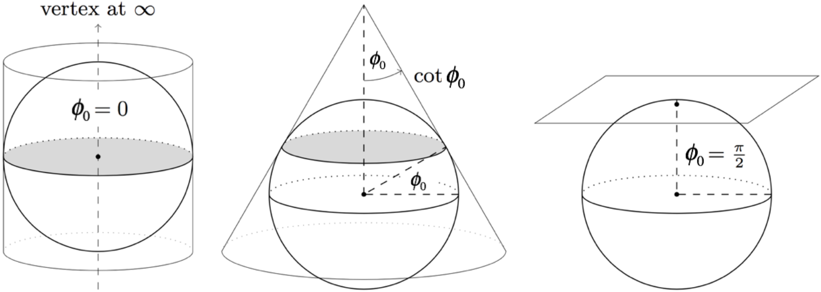

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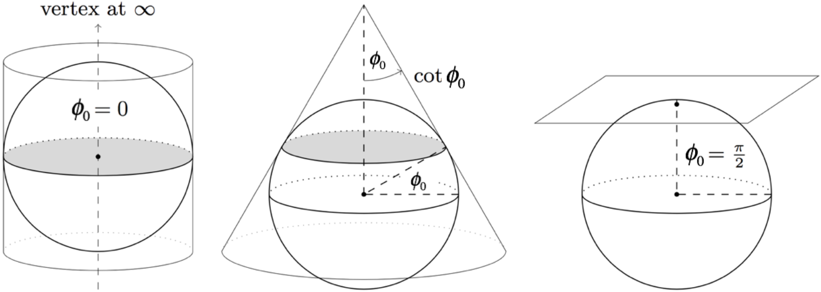

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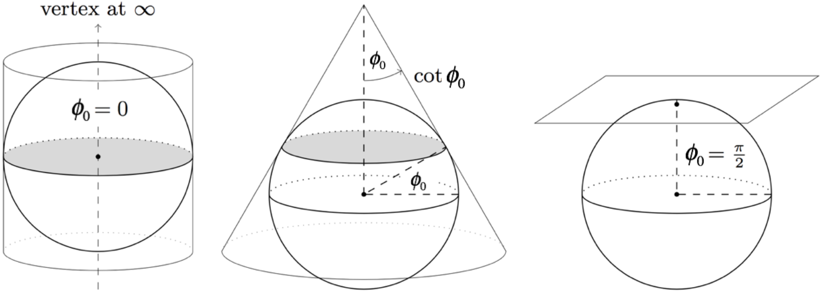

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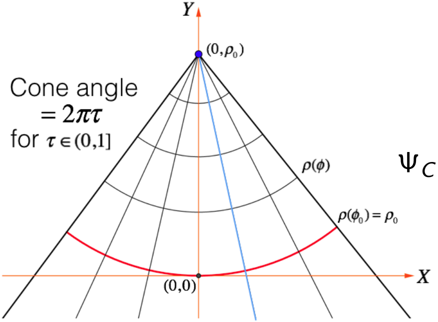
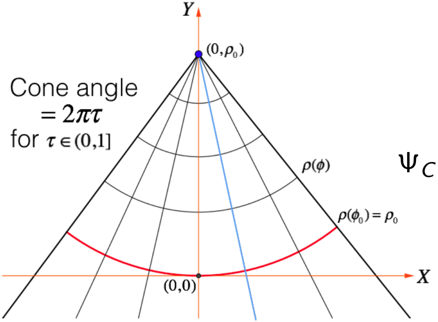
Let's cut open the cone and flatten it. How to specify the map?

General conic projections



$$\Psi_C : \begin{cases} X = \rho(\phi)\sin(\tau\theta) \\ Y = \rho_0 - \rho(\phi)\cos(\tau\theta) \end{cases}$$



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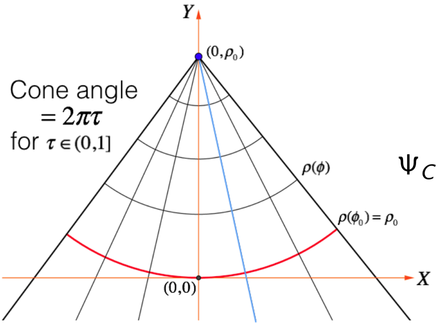

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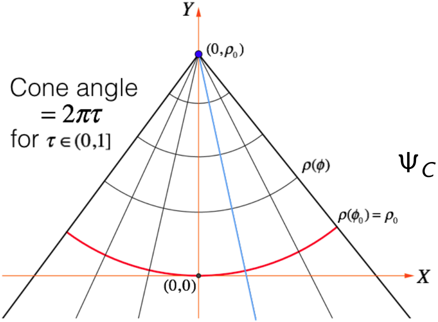

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- Require $\rho(\frac{\pi}{2}) = 0$ and $\rho'(\phi) < 0$.

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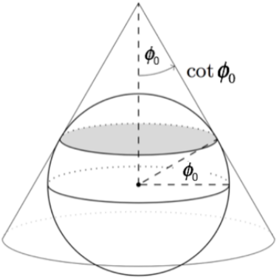
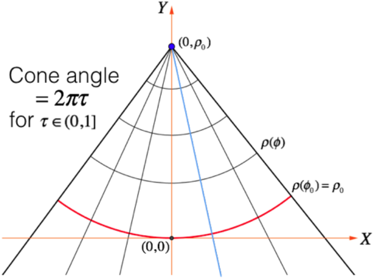
Still have several parameters to play with: $\boxed{\rho_0, \tau, \phi_0}$.

Conical maps





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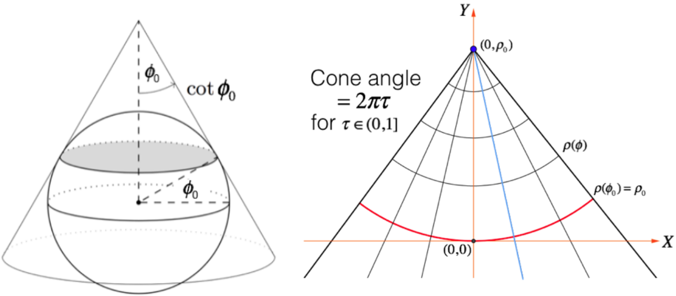

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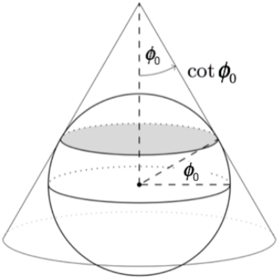

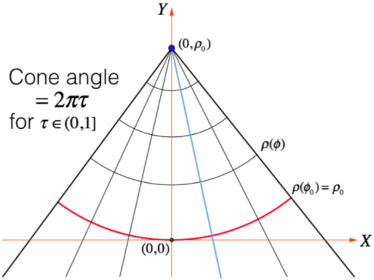

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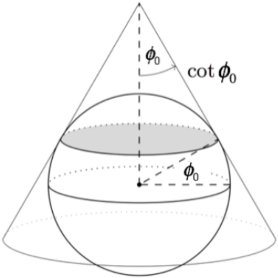

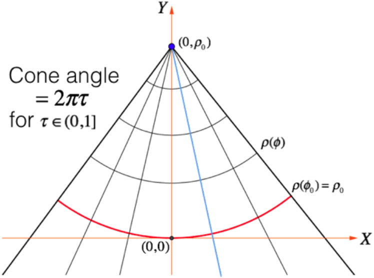

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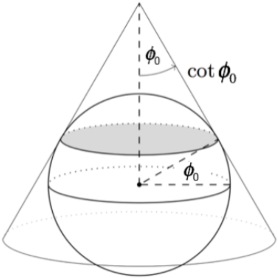

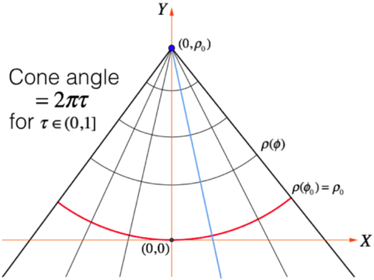

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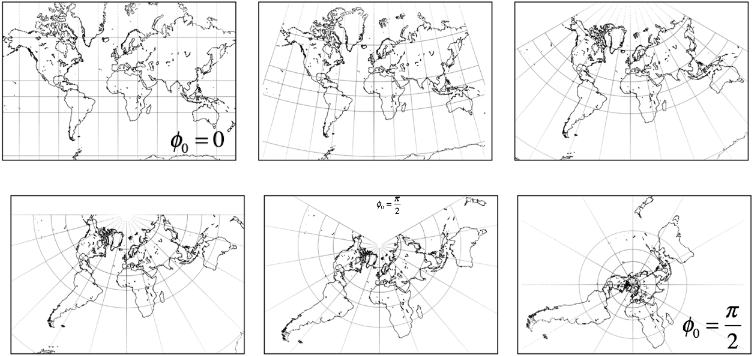

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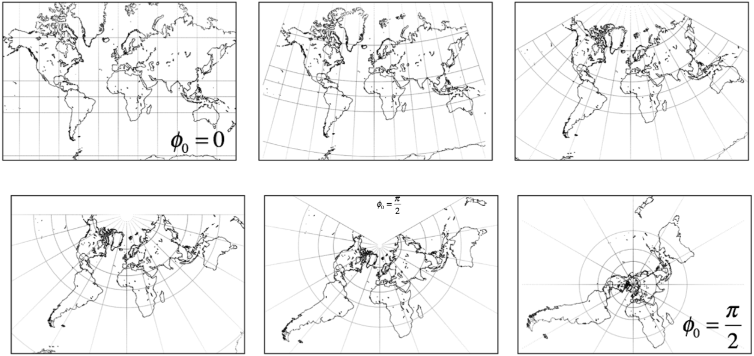

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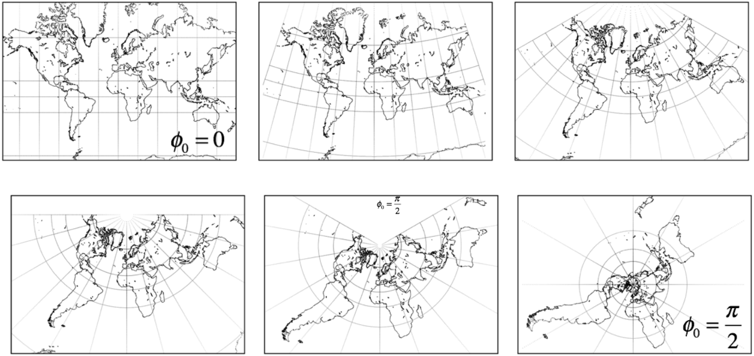

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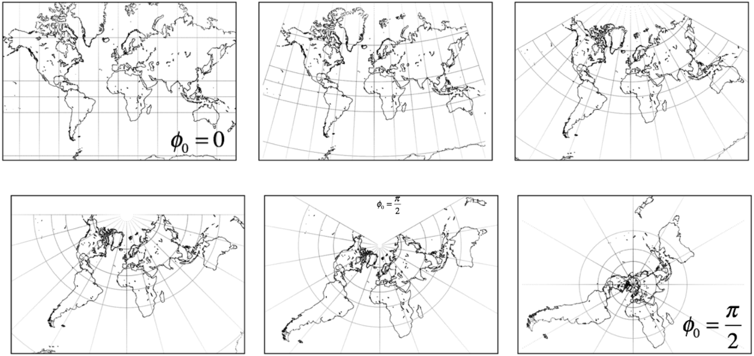

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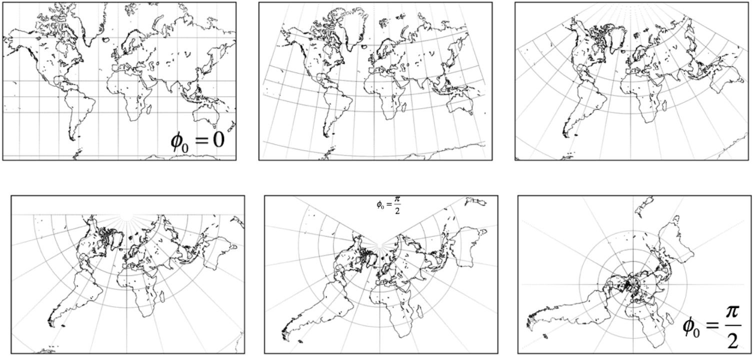

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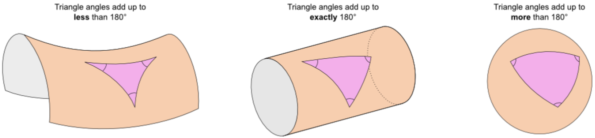

Negative Curvature *Zero Curvature* *Positive Curvature*

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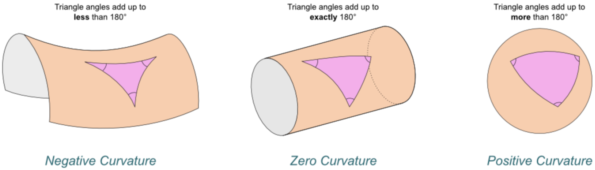

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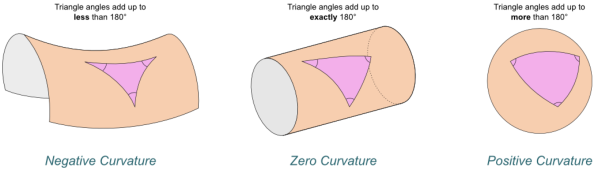

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Higher dim? Different structures? $\rightsquigarrow$ Differential geometry!